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Diseño de observadores no lineales en presencia de retardos desconocidos

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Resumen

En esta tesis se presenta el diseño de observadores para diferentes clases de sistemas no lineales. Estos sistemas presentan diferentes problemas relacionados con el muestreo y los retardos de tiempo. En primer lugar, se presenta una clase de sistemas no lineales no uniformemente observables con múltiples entradas y múltiples salidas, para esto se diseña un observador cuyo término de corrección se obtiene a partir de la solución de una ecuación diferencial ordinaria de Lyapunov. El observador diseñado es capaz de obtener la estimación del vector de estado actual continuo, teniendo en cuenta que el sistema original presenta múltiple muestreo y múltiple retardo en las señales de salida disponibles. La segunda clase de sistemas no lineales abordada en esta tesis es una clase de sistemas uniformemente observables; para esta clase de sistemas, se diseñan dos observadores con estructuras en cascada, cada uno para lidiar con diferentes problemáticas. El primer observador está diseñado para lograr la estimación del vector de estado actual junto con el valor del retardo desconocido presente en la salida del sistema. El segundo observador está diseñado para obtener la estimación del vector de estado real de un sistema no lineal con retardos presentes en la salida y en la dinámica del sistema. La particularidad de este diseño radica en las condiciones de transformación del sistema necesarias, ya que se considera que los retardos en el vector afín pueden estar presentes en diferentes estados y no necesariamente en todo el vector. La eficacia y las propiedades clave de los observadores propuestos se demuestran mediante su aplicación a ejemplos académicos, un péndulo simple, un brazo robótico y el sistema de infección por VIH.

Palabras clave: Sistemas no lineales, Observadores de alta ganancia, Observador en cascada, Sistemas muestreados, Sistemas con retardo, Infección por VIH, Brazo robótico, Péndulo simple.

Abstract

In this thesis, the design of observers for different classes of nonlinear systems is presented. These systems present different problems related to sampling and time-delays. First, a class of multiple-input/multiple-output nonlinear non-uniformly observable systems is presented, and an observer is designed whose correction term is obtained from the solution of a Lyapunov ordinary differential equation. The designed observer is able to obtain the estimation of the continuously actual state vector, considering that the original system has multiple sampling and multiple time-delays in the available output signals. The second class of nonlinear systems addressed in this thesis is a class of uniformly observable systems; for this class of systems, two observers with cascade structures are designed, each to deal with different problems. The first observer is designed to achieve the estimation of the actual state vector in conjunction with the unknown delay present at the output of the system. The second observer is designed to obtain the estimation of the actual state vector of a nonlinear system with delays present in the output and in the dynamics of the system. The particularity of this design lies in the necessary system transformation conditions, since it is considered that the delays in the affine vector can be present in different states and not necessarily in the whole vector. The effectiveness and key properties of the proposed observers are demonstrated through application to academic examples, a simple pendulum, a robotic arm, and the HIV infection system.

Keywords: Nonlinear systems, High gain observer, Cascade observer, Sampled systems, Time-delay systems, HIV infection, Robotic arm, Simple pendulum.

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List of publications

International journals

1. Hernández-González, O., Ramírez-Rasgado, F., Farza, M., Guerrero-Sánchez, M.-E., Astorga- Zaragoza, C.-M., M′Saad, M., and Valencia-Palomo, G. (2024). Observer for Nonlinear Systems with Time-Varying Delays: Application to a Two-Degrees-of-Freedom Helicopter. Aerospace 2024, 11, 206.

2. Ramírez-Rasgado, F., Farza, M., Hernández-González, O., M'Saad, M., and Astorga-Zaragoza, C.-M. (Submitted). Cascade Observer Design for a Class of Disturbed Nonlinear Systems with State and Output Delays. IEEE Transactions on Automatic Control.

3. Ramírez-Rasgado, F., Hernández-González, O., Farza, M., M'Saad, M., Astorga-Zaragoza, C.-M. and Guerrero-Sánchez, M.-E. (Submitted). Filtered Extended Observer for Building-like structure Under Disturbed Sampled Outputs. Journal of Vibration Engineering & Technologies.

4. Ramírez-Rasgado, F., Astorga-Zaragoza, C. M., Hernández-González, O., Guerrero-Sánchez, M. E., Osorio-Gordillo, G. L., and Reyes-Reyes, J. (2022). Observer synthesis for uncertain nonlinear systems with nonuniformly sampled and delayed output. IEEE Systems Journal, 16(3), 4532-4540.

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1. Ramírez-Rasgado, F., Hernández-González, O., Farza, M., M'Saad, M., Astorga-Zaragoza, and C.-M. (2023, December). Observer Design for a Class of Nonlinear System with an Unknown Time-delay in the Output. In 2023 IEEE 11th International Conference on Systems and Control (ICSC) Sousse, Tunisia, 2023, (pp. 189-194). IEEE.

2. Ramírez-Rasgado, F., Farza, M., Hernández-González, O., Astorga-Zaragoza, C. M., Valencia-Palomo, G., and Guerrero-Sánchez, M. E. (2023, October). New Cascade Observer for a Class of Nonlinear Systems with Time-delayed Outputs. In 2023 20th International Conference on Electrical Engineering, Computing Science and Automatic Control (CCE) Mexico City, Mexico, (pp. 1-4). IEEE.

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Chapter 1

General introduction

Over time, with the advancement of technology, the automatic control community has become increasingly interested in the estimation of system states. In an ideal world, engineers and scientists would have access to every internal variable of a dynamic system with infinite precision and zero delay. Unfortunately, this is not possible. In most practical scenarios − from biomedical systems and chemical reactors to robotics and beyond − we are left guessing the full internal behavior of a system using only a few, often noisy, and imperfectly sampled measurements. In view of this, algorithms known as state observers have been developed within the control community. These are useful for many purposes such as: monitoring of unknown variables, estimation of unknown parameters and/or dynamics, or the implementation of advanced control strategies for linear or nonlinear systems.

State estimation primarily involves inferring the internal dynamics of a system using the available inputs and outputs. This gives rise to different physical constraints and dynamic behaviors that give rise to different types of systems and must be taken into account when designing state observers. Observer designs have been developed to address these challenges and can be broadly categorized into model-free and model-based approaches, as suggested by Besançon (2007).

The first type is based on the nonlinear system approximation in which the mathematical model representing the system is not needed; some of the types are observers based on neural networks, fuzzy logic, and data-driven, to mention a few (Zhang and Wang, 2021; Cui et al., 2023; Alhajeri et al., 2021; Marani et al., 2025). The disadvantage of this type of observers is the region of application and the number of data that are necessary for its development.

The second type is based on the knowledge of the mathematical model of the system. Within this class, there are mainly two types that have been pioneers for the development of state observers. These observers are the Luenberger observer (Luenberger, 1964) and the Kalman filter (Kalman, 1960), which were designed for linear systems. However, most real systems exhibit inherently nonlinear behavior, so more sophisticated approaches are needed. Nowadays, within this type of observers are sliding mode approaches (Surjagade et al., 2020; Zhirabok et al., 2021; Chen et al., 2021b), generalized observers (Guo et al., 2020; Ortega et al., 2021) and high gain observers (Deza et al., 1992; Gauthier and Kupka, 1996) to name a few. The advantage of this class of observers is that, knowing the model of the system, the observers can be designed to cope with the different phenomena encountered at their inputs and outputs ensuring their convergence.

Although the convergence and robustness properties of the observers are strictly related to the observability properties of the underlying system. Observability in nonlinear systems is not a trivial matter and sometimes depends on the trajectories of the system. Roughly speaking, systems can be classified according to their observability as uniformly observable and non-uniformly observable. The class of the system has a significant impact on the design and performance of the observer. Uniformly observable systems guarantee the observability of the system for all time, since the state-affine matrix does not depend on either states or inputs. However, to bring it to this representation, it is necessary to transform the original system, which results in very complex transformations. In observer design for these systems, the observer gain is fixed and is obtained by different methods such as the solution of Linear Matrix Inequalities (LMI) that result in solutions of Riccati or Lyapunov algebraic equations. Some authors dealing with these systems are (Guerra et al., 2015, 2017; Nath et al., 2019; Tan and Edwards, 2001; Mohamed et al., 2012; Zemouche and Boutayeb, 2013; Zhang et al., 2012; Phanomchoeng and Rajamani, 2010; Farza et al., 1997; Kazantzis and Kravaris, 1998). It is important to note that although the LMI-based approach provides a systematic design for obtaining observer gain, this is not without limitations. One of the main disadvantages is that the nonlinear functions of the system satisfy certain conditions such as being globally Lipschitz or one-sided Lipschitz. These properties limit the growth or dynamic behavior of the nonlinearities. In order to obtain the gains it is necessary to know *a priori* the value of the Lipschitz constants. In practice, verifying or estimating these values is a challenge, especially for complex or poorly known systems, and this can result in overly conservative values.

Moreover, although LMIs are attractive due to their convexity, another problem often faced at the time of obtaining the gains is the involvement of Bilinear Matrix Inequalities

(BMI). These inequalities are intrinsically non-convex, more computationally demanding and can lead to infeasible or suboptimal solutions. Taking into account these challenges, the use of other types of observers, such as high gain observers, has been promoted. This observer avoids the knowledge of Lipschitz constants and complex optimization procedures. These observers are based on local observability and provide fast convergence and robustness close to an operating point, as shown in (Deza et al., 1992; Gauthier and Kupka, 1996; Esfandiari and Khalil, 1992).

Considering that Multiple-Input/Multiple-Output (MIMO) systems make the challenge of observer design even more challenging. These systems often lack the canonical observable structure that facilitates the design of Single-Input/Single-Output (SISO) systems. In particular, the presence of cross-couplings between inputs and outputs can complicate the observability properties, and the system may not be fully observable from any initial trajectory. Although classical high-gain and LMI-based observers provide powerful tools in the SISO context, their direct application to MIMO systems is often limited by the structural complexity of the model and the computational burden associated with high-dimensional gain matrices or Riccati equations. To deal with this challenge, different authors have extended and developed techniques to apply the high-gain approach to MIMO systems, in which no square structures or decoupled estimation strategies are used. Some authors who address this problem are (Farza et al., 2017a, 2011; Oueder et al., 2012; Gauthier and Kupka, 2001; Besançon, 2000; Hammouri et al., 2010; Lee and Park, 2003). For example, Farza et al. (2017b) proposed an extended high-gain observer that adapts to systems without triangular or canonical structures. Similarly, Farza et al. (2011) and Oueder et al. (2012) introduced observer designs that relax the structural triangularity requirement by allowing dynamic gain adaptation and estimation strategies suitable for systems with interdependent dynamics. In Gauthier and Kupka (2001) and Besançon (2000), techniques for designing high-gain observers in MIMO systems by exploiting observability distributions and immersion techniques are presented. In addition, a high gain observer for multi-output structured nonlinear systems is presented in Hammouri et al. (2010), which offers convergence guarantees under wider observability conditions. These approaches demonstrate that, even in the presence of structural and dimensional complications, high-gain observers are still very valuable due to their simplicity, strong convergence properties, and the ability to handle partially known dynamics or bounded uncertainties.

In contrast to uniformly observable systems, where the observability condition is satisfied globally and independently of the input, non-uniformly observable systems present significant challenges for observer design. In these systems, observability may depend on specific input trajectories or be satisfied only in certain regions of the state space, complicating the derivation of observers with global convergence guarantees. This phenomenon is particularly prevalent in nonlinear systems, biological models and complex MIMO configurations, where the system may exhibit partial or transient observability depending on its operating scheme. For example, Besançon and Hammouri (1996) proposed an observer design strategy for non-uniformly observable systems, exploiting rank conditions on the observability distribution and applying immersion techniques to construct observers that ensure local convergence. They showed that even when the system does not satisfy uniform observability globally, it is still possible to reconstruct the state along system trajectories if sufficient observability information is available locally. Their approach extends the applicability of nonlinear observers to a broader class of systems by moving beyond traditional assumptions such as involutive or full-rank globally. Subsequently in Hernández-González et al. (2016) and Farza et al. (2017b) high gain observers for MIMO systems have been presented without requiring canonical forming, relying instead on weaker local observability assumptions. In many of these works, observer convergence is no longer guaranteed solely by uniform observability, but rather by the presence of persistently exciting signals, which ensure that sufficient information is injected into the system over time to allow state reconstruction. This condition is known as the persistent excitation condition and requires that the input or output trajectories provide sufficiently rich excitation over finite time intervals. For instance, Gautheir et al. (1992) and Kazantzis and Kravaris (1998) showed that, in systems lacking global or uniform observability, the persistent excitation condition can guarantee asymptotic convergence of the estimation error. Taking this into account, in Farza et al. (2011, 2017b) this principle was applied to design high-gain and extended observers, showing that, under persistent excitation, convergence can still be achieved even in the absence of triangular shapes or full-rank observability at the global level.

Another example where this condition is exploited is presented in Torres et al. (2011), where the challenge of designing observers for a class of nonlinear systems, including a matrix affine to state dependent on the output is addressed. An observer similar to the extended Kalman filter (EKF) adapted to systems that can be expressed as a linear part in the unmeasured states and an additive nonlinearity with a triangular Jacobian is proposed. The authors showed that, under the persistent excitation condition, the observer guarantees exponential stability of the estimation error. These contributions reflect a shift toward observer designs capable of operating under realistic, uncertain and structurally constrained conditions, moving beyond classical assumptions of uniform observability and increasingly emphasizing the role of signal excitation as a practical substitute for stringent structural requirements.

Starting from theoretical developments and practical limitations of observer design for nonlinear and nonuniformly observable systems, this thesis aims to address a set of challenges that remain largely open in the literature. Although significant progress has been made in addressing structural constraints, partial observability, and excitation-dependent convergence, real-world systems often present additional complexities that complicate the implementation of reliable state estimation. A fundamental assumption in most observer designs is that the inputs and outputs of the system are known accurately, continuously, and synchronously. However, due to physical and technological limitations, this assumption often fails in practice. Measurement devices may introduce delays, noise, asynchronous sampling, or limited resolution, while some signals may not be accessible in real time or arrive with unknown latencies. These problems introduce instrumentation-related challenges that significantly affect the performance and even the feasibility of observer-based strategies, especially in nonlinear systems operating under limited sensing conditions.

1.1 Objectives

1.1.1 Main objective

Estimate continuously the actual state vector for different classes of nonlinear systems with time-delayed outputs (known and unknown), which are only available at instants of time through the design of nonlinear observers.

Consequently, this work focuses on three interrelated and technically challenging problems:

> **The presence of multiple sampling instants and multiple known time-varying delays in the system outputs.** This invalidates the classical assumptions of synchronous measurements.
>
> **The presence of unknown delay in the output of the system.** This adds uncertainty to the available measurement of the system and addresses a case that occurs in practice.

The existence of time delays in states and output system where triangularity is not necessarily respected. This addresses the inherent problem of physical phenomena in conjunction with data transmission phenomena.

Taking into account the above, the specific objectives of the work can be defined.

1.1.2 Specific objectives

- Estimate continuously the actual state vector of a nonlinear nonuniformly observable system whose output is multi-sampled and multiple time-varying using a high-gain observer.
- Jointly estimate the actual state vector and the value of the unknown delay for a uniformly observable system whose output presents an unknown time delay using a cascade observer.
- Estimate the actual vector for a uniformly observable system exhibiting state and output delays using a cascaded observer.

These scenarios are not only relevant from a theoretical point of view, but also arise naturally in multiple types of systems, networks with multiple sensors of different natures, and control applications subject to communication constraints.

1.2 Contributions

The first contribution of this work consists in the design of an observer for a class of non-uniformly observable MIMO systems whose outputs are acquired at multiple time instants and are affected by multiple time-varying delay functions. In this contribution, the estimation of the continuous and actual state vector of the system is sought. The dynamics of the nonlinear system can be seen in two parts. The first is the state affine part, in which it is considered that in the state affine matrix are present states and inputs of the system, and the second part is related to the nonlinear part of the system, which presents a triangular structure with the state. The main motivation to address this sampling and delay problem arises because, in practice, it is necessary to have multiple sensors of different natures to have physical redundancy of critical variables.

Different authors have addressed problems related to sampling and delay such as (Germani et al., 2002; Karafyllis and Kravaris, 2009; Farza et al., 2015b; Nadri and Grajales, 2013;

Hernández-González et al., 2016; Farza et al., 2018a; Ling and Kravaris, 2019; Tréangle et al., 2019; Hernández-González et al., 2021; Ramírez-Rasgado et al., 2022a,b; Zhuang et al., 2024, 2023; Cacace et al., 2014a; Shen et al., 2017; Jaramillo et al., 2021; Ahmed-Ali et al., 2013; Chen et al., 2021a; Feddaoui et al., 2020) and so on.

These problems can be addressed in the following ways, sampling, delay and both. First, in the case of sampled outputs, one of the most common approaches is to model the system as a switched system, where one dynamic model is used when the output is available and another when it is not. This idea is explored in works such as (Jaramillo et al., 2021; Ahmed-Ali et al., 2013; Chen et al., 2021a; Feddaoui et al., 2020), where impulsive or hybrid observer structures are proposed. Alternatively, some authors have designed dynamic compensators that allow using a unified observer structure regardless of whether the output is available. In Hernández-González et al. (2021) a high-gain observer is proposed for nonlinear MIMO systems with output measurements available at irregular time instants. Unlike switched observers, the proposed structure includes a dynamic correction term that adapts whether the output is present or not, allowing a single observer law to handle both situations. In (Adil et al., 2022) this Linear Parameter-Varying (LPV) approach is used in the same way for the design of the observer; however, in this article, a small delay value is proposed so it is not necessary to use the cascade to estimate the actual state vector.

Second, in systems with delayed outputs, another set of observer strategies has been proposed. These include designs with internal predictors (Karafyllis and Kravaris, 2009), as well as cascade structures, where an auxiliary subsystem reconstructs the delayed output before it is used by the main observer (Germani et al., 2002; Cacace et al., 2014b; Hernández-González et al., 2016). In this direction, Karafyllis and Kravaris (2009) developed a predictor-based observer for nonlinear systems with known time-varying output delays, ensuring convergence by constructing a Lyapunov-Krasovskii functional. Their framework addresses both constant and time-varying delays and provides rigorous global asymptotic stability guarantees under the assumption that the delay is measurable and its derivative is bounded, conditions that, although restrictive, allow an accurate theoretical treatment of the dynamics of the estimation error.

A seminal contribution in this direction is that of Germani et al. (2002), who proposed a cascade observer for linear systems subject to a known constant output delay. The strategy involves a prediction stage that reconstructs the delayed measurement using internal dynamics, which is then passed to a standard Luenberger observer thereby restoring the structure expected by classical designs and enabling convergence despite the presence of delayed information. Similar ideas have been extended to nonlinear systems, where delay compensation precedes the state estimation stage, often under the assumption of delay bounds or full delay knowledge. These designs typically aim to compensate for the effect of output delay through structured prediction, ensuring convergence of the estimation error so long as the delay behaves itself and the system satisfies the required observability properties. Meanwhile, Cacace et al. (2014b) introduced a cascaded observer for systems with output delays and nonlinearities. Their method includes a dynamic predictor that reconstructs the delayed measurement and a principal observer that estimates the state using the compensated signal. A key feature of their design is the use of generalized triangular structures, which facilitate the separation of the delay processing and state estimation stages, improving modularity and implementation. On their side, Hernández-González et al. (2016) addressed nonlinear MIMO systems with delayed outputs by proposing a cascade high-gain observer. In their framework, the first layer estimates the delayed output using an internal model and available measurements, while the second layer applies a high-gain correction to estimate the full state vector. This design is especially effective in systems that are not uniformly observable and where the output delay is known and potentially large, but bounded. In all of these contributions, the common thread is the assumption that the delay is known or bounded, and that the observer can be structured to absorb its effect before applying classical estimation techniques.

Finally, in more realistic scenarios, the system may exhibit both output delays and non-uniform sampling, a combination that increases the complexity of the observer design and undermines most classical assumptions. In such contexts, observer structures must simultaneously address sampling and output delay. Some recent work has taken up this challenge by developing hybrid strategies that combine delay compensation with sampling correction mechanisms. This is in order to achieve continuous and actual state estimation of the system. In Ramírez-Rasgado et al. (2022a) an observer for uncertain nonlinear systems with non-uniformly delayed and sampled outputs is proposed. Their approach introduces a correction term that is triggered only when new data are available, while an internal model maintains the evolution of the observer during the intervals between samples. This allows the observer to remain consistent even in the presence of delayed measurements and irregular sampling intervals. Ramírez-Rasgado et al. (2022b) explored how the structure of a high-gain observer can be redesigned depending on the nature and availability of the output signals, whether delayed, sampled, or both. Similarly, Zhuang et al. (2023) proposed a sampled data observer for nonlinear systems with bounded output delays, where the observer employs a prediction-based mechanism to synchronize internal estimation with delayed and discretely available outputs. Their design incorporates bounding conditions and structural observability properties to ensure convergence despite the asynchronous and delayed nature of the measurements. In (Tréangle et al., 2019), the base observer is modified by adding a dynamic function that compensates for a small delay value. Their method is designed for non-uniformly sampled outputs with known delays. Another approach is to use sliding modes for the design of the base of the cascade to achieve the synchronization of chaotic systems in (Hamoudi et al., 2020). Therefore, these approaches demonstrate the need for observer architectures that account for both delays and sampling, conditions that are very common in sensor networks, biomedical systems, and real-time control applications.

The second contribution of this thesis work consists of designing an observer that is able to estimate the current state vector for a uniformly observable system in the presence of unknown time delay at the output of the system. In this contribution primarily two things are sought, the estimation of the unknown delay value and the estimation of the actual state vector. The motivation for this contribution lies in a still open problem within the communication problems in the systems, the unknown delays. These unknown delays may be due to a variety of real-world factors that are common in engineering systems. For instance, asynchronous sampling of sensors can result in inconsistent synchronization between signal acquisition and observer processing. In networked control systems or wireless sensor networks, delays are often due to unpredictable transmission times due to congestion, interference or packet retransmission. In addition, internal preprocessing of smart sensors, such as digital filtering, calibration or buffering, can introduce variable delays that are not explicitly reported. Other causes are the lack of a shared time base between distributed measurement units, as well as system-level issues such as processor overhead or latencies induced by multitasking on embedded platforms or the number of networked elements.

Some proposals for estimation have been presented in (Ghanes et al., 2016; Gamiochipi et al., 2016; Shen et al., 2022). In Ghanes et al. (2016), an adaptive observer is proposed, which is designed with the maximum delay present in the system; convergence to zero is only assured when the delay in the system is equal to the delay in the observer. An extension of this work for a more general class is presented in Gamiochipi et al. (2016). In Shen et al. (2022), using a cascade observer, the actual vector can be estimated with an error equal to the difference between the real delay and the set delay in the observer. In the base, an observer based on sliding mode is proposed, and the predictors are a traditional structure. However, designing observers based on the maximum expected delay introduces several limitations. First, it can lead to performance degradation when the actual delay is less than the assumed delay, as the observer reacts slower than necessary. Second, in systems where the delay is time-varying or uncertain, fixing the observer to the worst-case delay can lead to persistent estimation errors or overly conservative dynamics. In addition, these designs often require prior knowledge or accurate estimation of the maximum delay, which is not always feasible in practice.

Other authors have focused on the estimation of the unknown delay in order to achieve the estimation of the actual state vector; some approaches have been presented Cacace et al. (2016); Zheng et al. (2018); Léchappé et al. (2018). In Cacace et al. (2016), an observer is proposed which is coupled to an identifier. The state vector is estimated by a high gain observer and the delay is estimated by an identifier. In Zheng et al. (2018), it introduces a function that hides the unknown delay, and by means of a sliding mode observer, it achieves the estimation of this function and reconstructs the delay. In Léchappé et al. (2018), the state is extended to estimate the states and the delay present at the input. First, it approximates the variable where the delay occurs using Taylor's theorem and its derivatives are obtained with Levant's differentiator; the unknown time-delay is obtained using a projection operator. In Nguyen et al. (2021a), an observer is presented for a nonlinear system with unknown and bounded delay, present at the input. In this work, the system is rewritten and the delay is represented in a new function; the observer by sliding modes assumes this new function as a disturbance, which is estimated by the observer. Subsequently, for the delay calculation, it uses this delay-estimated signal. An extension of this work using the LPV approach is presented in Nguyen et al. (2021b). It is important to note that although these methods are intended to provide delay-free state estimation, they have limitations. Approaches based on sliding or differentiating modes, high sensitivity to measurement noise can degrade the accuracy of the estimate. Methods that rely on auxiliary functions or identifiers to reconstruct the delay introduce more complexity and possible mismatches between the actual dynamics and the internal model. Also, the computational demands can be challenging for real-time application, especially when higher-order derivatives or projection operators are involved. Therefore, these factors propose a trade-off between adaptability and robustness of delay estimation strategies.

The final contribution of this thesis focuses on the design of an observer for uniformly observable systems that exhibit both output delays and delays embedded within the nonlinear structure of the system. This contribution aims to estimate the actual state vector even in the presence of delays within the state-affine dynamics that do not satisfy triangularity conditions. The motivation lies in addressing two critical sources of delay: those introduced by instrumentation constraints and those inherent to the nonlinear system behavior. Several authors have studied related problems (Hernández-González et al., 2024; Di Ferdinando et al., 2023; Vafaei and Yazdanpanah, 2016; Talebi et al., 2020).

In Di Ferdinando et al. (2023), an observer-based controller is designed for a state-affine system with sampled inputs and outputs, as well as delays in the system states. The observer relies on Sontag's universal formula to estimate the state vector and enable system control. Depending on the sampling and delay values, different observer configurations are derived. A notable aspect of this approach is that the observer provides a discrete-time estimate of the state vector. Sufficient conditions for practical stability are established.

In Hernández-González et al. (2024), the authors propose an observer for a non-uniformly observable nonlinear system with output sampling and delayed input. The design is based on a redesign of the observer proposed by Tréangle et al. (2019), achieving continuous state estimation while compensating for output sampling effects. Using a Lyapunov-Krasovskii framework, convergence of the observer is guaranteed and shown to depend on the sampling characteristics. However, these approaches consider only short output delays; when delays increase, cascade observer structures become necessary.

Following the line of uniformly observable systems, in Vafaei and Yazdanpanah (2016), the authors present the design of a cascaded observer, which replicates the structure of the observer basis for all predictors. In this work, the nonlinear system is considered to have delays in both inputs and states. Using a LPV approach, we obtain a series of LMIs for all the subsystems, which when solved we obtain the gain matrices for each subsystem. The main limitation of this work is in the dimensioning of the matrices to be solved, since it depends on the number of predictors in the cascade. A different strategy is presented in Talebi et al. (2020), where the observer gains are tuned using pole placement rather than relying on LMI-based formulations. This approach offers greater flexibility in the design process and ensures exponential convergence of the estimation error, provided that a sufficient number of predictor stages is incorporated into the cascade structure.

This work is composed of 5 chapters, from the introduction to the conclusions and two Appendices A and B.

The second chapter is devoted to the design of a high gain observer for nonlinear non-uniformly observable MIMO systems. The particularity of this observer is the ability to achieve the estimation of the current state vector considering that each of the outputs has multiple sampling intervals and multiple time-varying delay functions. In order to better understand the design, we first present the design considering that the outputs have single sampling intervals and the single time-varying delay functions. The persistent excitation condition that guarantees observability is formally presented. The case for uniformly observable systems has been addressed in Ramírez-Rasgado et al. (2021). Subsequently, the observer is redesigned to be able to deal with multiple sampling and multiple delays. In both cases, it is shown through convergence analysis that the convergence error converges exponentially to zero and the rate of convergence depends on the maximum values of delay and sampling. The performance of the observer is shown by considering the biological system of HIV infection in which the estimation of the unknown state variables of the system is achieved, and exploiting the property of non-uniformly observable systems, the estimation of unknown parameters is achieved.

The third chapter focuses on the design of an observer for SISO (Single-Input /Single-Output) uniformly observable nonlinear systems with unknown time-delay in the output. For this case, the cascade structure shown in Germani et al. (2002); Farza et al. (2018b); Hernández-González et al. (2016); Tréangle et al. (2018) is used, but the observer head is redesigned in order to cope with the unknown time-delay. To achieve this, the design of the cascade head is presented first, which in this case will not only be in charge of the estimation of the delayed vector of the system, but it must also estimate the unknown value of the delay. For this, it is proposed to rewrite the system in its delayed version, which causes the delay to remain only at the input of the system. Since the delay is an unknown parameter, the new system falls back to a non-linear parameterization. With this in mind and inspired by the proposal in Farza et al. (2014a) an adaptive observer is proposed, achieving the estimation of the state vector that is the same as the vector with delay of the original system and the unknown value of the time-delay present at the output. The formulation of the persistent excitation condition that assures the estimation of the unknown parameter is also presented. Subsequently, with the previous time-delay estimation, the predictors (m) in the cascade manage to compensate for the time-delay in the system in a fraction equal to $\hat{\tau}/m$, where the last predictor achieves the estimation of the current vector of the system. It is shown through convergence analysis that the observation error of each subsystem and of the unknown parameter converge exponentially to zero. Achieving the objective of joint estimation of the value of the unknown time-delay and the actual state vector. The performance of the proposed observer has been illustrated through an academic example and application to a robotic arm.

The fourth chapter deals with the design of an observer for an uncertain uniformly observable SISO systems with delays in the output and in the system states. The particularity of this system is that it allows the presence of delays in the affine vector and they are not limited only to the nonlinear part of the system. As approached by some authors (Di Ferdinando et al., 2023; Vafaei and Yazdanpanah, 2016; Talebi et al., 2020). To achieve this, the cascade observer is redesigned to deal with the presence of delays in states. First, the simple case, delay in the output and delay in the nonlinear function, is presented to properly present the redesign of the cascade observer. Subsequently, the case where the affine state vector has delays is discussed. Subsequently, the case where the affine state vector has delays is discussed. Through a change of variables, the triangularity of the system is achieved and the previously designed observer is used to achieve the estimation of the actual state vector. Likewise, the methodology to reconstruct each one of the state variables considering if the delay is commensurable or non-commensurable is presented. It is shown through convergence analysis that the observation error converges exponentially to a bounded region near the origin in the presence of uncertainty and to zero if the system is free of uncertainties. The performance of the proposed observer has been illustrated through an academic example and a simple pendulum.

This thesis presents the conclusions obtained from the results of this work and some perspectives for future work.

Finally, in Appendix A, some theoretical notions of nonlinear systems, observability conditions and different classes of systems are presented. In Appendix B the lemmas used in chapters 2, 3 and 4 are presented.

Chapter 2

Observer synthesis for non-uniformly observable nonlinear systems with multi-rate sampled and delayed output

2.1 Introduction

In this chapter, the design of an observer for a class of nonlinear non-uniformly observable systems, whose output presents sampling and time-varying delays. This class of system allows the design of observers considering coordinate changes less restrictive than other approaches. However, the condition of persistent excitation in the system has to be fulfilled to satisfy the necessary observability conditions.

The methodology for the design of an observer for state estimation in a practical case is presented. It is considered that the system has multiple outputs and each output has different sampling and delays, which increases the observation problem. First, in order to introduce necessary notation, the case of a simple sampling and a simple delay is addressed. Subsequently, it is extended to the multiple case, addressing a current and little investigated problem within this class of nonlinear systems.

The effectiveness of the observer is shown through simulations considering the HIV biological system. Different sampling cases and delays are shown, as well as the joint estimation of unknown states and parameters of the system.

2.2 Problem statement

Consider the following class of Multiple-Input Multiple-Output (MIMO) nonlinear systems:

$$\mathcal{SYS}: \begin{cases} \dot{x}(t) &= A\left(u(t), x(t)\right) x(t) + \varphi\left(u(t), x(t)\right) \\ y^a(t) &= C x_\tau(t) = x_\tau^{(1)}(t) \end{cases} \tag{2.1}$$

where

$$x = \begin{pmatrix} x^{(1)} \\ \vdots \\ x^{(q)} \end{pmatrix} \in \mathbb{R}^n \text{ where } x^{(k)} = \begin{pmatrix} x_1^{(k)} \\ \vdots \\ x_{n_k}^{(k)} \end{pmatrix} \in \mathbb{R}^{n_k}, k = 1, \ldots, q \text{ with } \sum_{k=1}^{q} n_k = n, \tag{2.2}$$

$$A(u(t), x(t)) = \begin{bmatrix} 0 & A^{(1)}(u(t), x^{(1)}(t)) & \ldots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & A^{(q-1)}(u(t), x^{(1)}(t), \ldots, x^{(q-1)}(t)) \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.3}$$

$$C = \begin{bmatrix} I_{n_1 \times n_1} & 0_{n_1 \times n_2} & \ldots & 0_{n_1 \times n_q} \end{bmatrix},$$

$$\varphi(u(t), x(t)) = \begin{pmatrix} \varphi^{(1)}(u(t), x^{(1)}(t)) \\ \vdots \\ \varphi^{(i)}(u(t), x^{(1)}(t), x^{(2)}(t), \ldots, x^{(i)}(t)) \\ \vdots \\ \varphi^{(q)}(u(t), x(t)) \end{pmatrix}, \tag{2.4}$$

where $x(t) \in \mathbb{R}^n$ and $x_\tau(t) \in \mathbb{R}^n$ are the actual and delayed versions of the system state variables, each $A^k(u(t), x(t)), k = 1, \ldots, q-1$, is a $n_k \times n_{k+1}$ non-null matrix which is triangular with respect to x, i.e.

$$A^{(k)}(u(t), x(t)) = A^{(k)}(u(t), x^{(1)}(t), ..., x^{(k)}(t)), \; k = 1, \ldots q-1,$$

the nonlinear vectorial function is denoted by $\varphi(x, u)$ and each component of which assumes a triangular structure with respect to x, i.e.

$$\varphi^{(k)}(u(t), x(t)) = \varphi^{(k)}(u(t), x^{(1)}(t), \ldots, x^{(k)}), k = 1, \ldots, q, \tag{2.5}$$

the input system is denoted by $u \in \mathbb{R}^m$ and $y^{(a)} \in \mathbb{R}^p$ denotes the available output that will be defined below in accordance with the output measurement availability.

Two design features are worth to be pointed out to appreciate the observer design problem under consideration. First, the system (2.1)-(2.4) is not uniformly observable. Second, the available output measurements are expressed as a function of the state variables in a delayed form, implying thereby that $n_1 = p$.

In order to continue it is important to define the following usual assumptions that are required in the high gain observer synthesis for non uniformly observable systems (Hernández-González et al., 2016).

Assumption 2.2.1 *The system input and state are bounded, i.e., there exist compact sets $U \subset \mathbb{R}^m$ and $X \subset \mathbb{R}^n$ such that $\forall t \geq 0,\ u(t) \in U$ and $x(t) \in X$, thus allowing one to define the upper bounds.*

$$\begin{cases} \rho_M = \max\limits_{\substack{1\leq k\leq q \\ 1\leq i\leq n_k}} \sup\limits_{t\geq t_0} \mid x_i^{(k)}(t) \mid,\ x_M \triangleq \sup\limits_{t\geq 0} \|x(t)\|, \\ \\ A_M \triangleq \sup\limits_{u\in U, x\in X} \|A(u(t),x(t))\|,\ \varphi_M \triangleq \sup\limits_{u\in U, x\in X} \|\varphi(u(t),x(t))\|. \end{cases} \tag{2.6}$$

Assumption 2.2.2 *Each entry of the matrices $A(u(t),x(t))$ and $\varphi(u(t),x(t))$ is Lipschitz on X with respect to x uniformly in u, i.e. $\exists L_A > 0$;$\exists L_\varphi > 0$; $\forall u \in U$; $\forall (x,\bar{x}) \in X \times X$, one has*

$$\|A^k(u(t),x(t)) - A^k(u(t),\bar{x}(t))\| \leq L_A \|x(t) - \bar{x}(t)\|,\ k = 1 \ldots q-1,$$

$$\textit{and}\ \|\varphi^k(u(t),x(t)) - \varphi^k(u(t),\bar{x}(t))\| \leq L_\varphi \|x(t) - \bar{x}(t)\|,\ k = 1 \ldots q.$$

Now, one recalls the following (usual) saturation function

$$\forall z \in \mathrm{I\!R},\ sat_r(z) = r \cdot sat\left(\frac{z}{r}\right) = \begin{cases} z & \text{if } \mid z \mid \leq r, \\ r \cdot sign(z) & \text{if } \mid z \mid > r, \end{cases} \tag{2.7}$$

where $r > 0$ is a non negative real and the signum function is given by $sign(\cdot)$. Similarly, one defines a vectorial version of this function as follows

$$Sat_r(x^{(k)}) = \begin{pmatrix} sat_r(x_1^{(k)}) \\ \vdots \\ sat_r(x_{n_k}^{(k)}) \end{pmatrix}. \tag{2.8}$$

Using these functions, one saturates the system nonlinearities $A^{(k)}\left(u(t), x(t)\right)$ and $\varphi^{(k)}$'s for $k = 1, \ldots, q$ as follows

$$\begin{aligned} A_s^{(k)}(u(t), x^{(1)}(t), \ldots, x^{(k)}(t)) &\triangleq A^{(k)}(u, Sat_R(x^{(1)}(t)), \ldots, Sat_R(x^{(k)}(t))), \\ \varphi_s^{(k)}(u(t), x^{(1)}(t), \ldots, x^{(k)}(t)) &\triangleq \varphi^{(k)}(u(t), Sat_R(x^{(1)}(t)), \ldots, Sat_R(x^{(k)}(t))), \end{aligned} \tag{2.9}$$

with

$$R > \rho_M, \tag{2.10}$$

where ρ_M is the positive real defined as in (2.6). Bearing in mind that the Lipschitz constant of the saturation function is equal to one, one gets

$$\begin{aligned} &\forall u \in U;\ \forall (x, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}^n, \\ &\|A_s^k(u(t), x(t)) - A_s^k(u(t), \bar{x}(t))\| \leq L_A \|x(t) - \bar{x}(t)\|, \\ &\|\varphi_s^k(u(t), x(t)) - \varphi_s^k(u(t), \bar{x}(t))\| \leq L_\varphi \|x(t) - \bar{x}(t)\|. \end{aligned} \tag{2.11}$$

Since the Lipschitz property (2.11) holds globally, the observer design will be achieved for the following class of systems

$$\mathcal{SYS} : \begin{cases} \dot{x}(t) &= A_s\left(u(t), x(t)\right) x(t) + \varphi_s\left(u(t), x(t)\right) \\ y^a(t) &= C x_\tau(t) = x_\tau^1(t) \end{cases} \tag{2.12}$$

where

$$\begin{cases} A_s(u(t), x(t)) &= \begin{bmatrix} 0 & A_s^{(1)}(u(t), x^{(1)}(t)) & \ldots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & A_s^{(q-1)}(u(t), x^{(1)}(t), \ldots, x^{(q-1)}(t)) \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ \varphi_s(u(t), x(t)) &= \begin{pmatrix} \varphi_s^{(1)}(u(t), x^{(1)}(t)) \\ \vdots \\ \varphi_s^{(i)}(u(t), x^{(1)}(t), x^{(2)}(t), \ldots, x^{(i)}(t)) \\ \vdots \\ \varphi_s^{(q)}(u(t), x(t)) \end{pmatrix}. \end{cases} \tag{2.13}$$

It is important to emphasize that due to the fact that system (2.12) coincides with system (2.1) on X, the observer that will be designed for system (2.12) could be used to predict the trajectories of system (2.1) since it always lies in X.

2.2.1 Output measurements availability

As it was mentioned above, the admissible output $y^a \in \mathbb{R}^p$ is specified according to an adequate multi-rate sampling process of its components which will be denoted y_i^a for $i = 1, \ldots, p$. Indeed, one assumes that each component of the output measurement y_i^a is available only at sampling time instants $t_i^0 < t_i^1 < \ldots < t_i^{k_i} < \ldots$ with time-varying intervals $\Delta_i^{k_i} = t_i^{(k_i+1)} - t_i^{k_i}$ satisfying

$$\lim_{k_i \to \infty} t_i^{k_i} = +\infty \quad \text{and} \quad t_i^0 = t^0 \quad \text{for } i = 1, \ldots, p.$$

Furthermore, each component of the output measurement y_i^a is obtained at the time instant $t_i^{k_i}$ with a time-varying delay denoted by $\tau_i(t_i^{k_i})$, leading thereby to the following expression of the available output

$$y_i^a(t) = y_i\left(t_i^{k_i} - \tau_i(t_i^{k_i})\right) \;\; \forall t \in \left[t_i^{k_i}, t_i^{(k_i+1)}\right[. \tag{2.14}$$

Three assumptions are made to further detail the availability of the output measurement.

Assumption 2.2.3 *The time instants associated to the measurements of the outputs, i.e.* $t_i^{k_i} - \tau_i(t_i^{k_i})$ *are positive and each sequence* $\left\{t_i^{k_i} - \tau_i(t_i^{k_i})\right\}_{k_i \geq 0}$ *for* $i = 1, \ldots, p$ *is increasing.*

Assumption 2.2.4 *All the sampling partition diameters are bounded, i.e.*

$$\begin{aligned} &\exists \Delta_M > 0;\ \forall i \in \{1, \ldots, p\}, \\ &0 < \Delta_i^{k_i} = t_i^{(k_i+1)} - t_i^{k_i} \leq \Delta_M \ \forall k_i \geq 0 \end{aligned} \tag{2.15}$$

Assumption 2.2.5 *All the delay functions* $d_i(t)$*, are non negative and bounded, i.e.*

$$\exists \tau_M > 0;\ \forall i \in \{1, \ldots, p\}; \forall t \geq t_0^0,\ 0 \leq \tau_i(t) \leq \tau_M. \tag{2.16}$$

First, the design of the observer will be presented considering the case of a single sampling partition with a single time-varying delay, then the case of multi-rate sampling with multiple time-varying delays.

2.3 Single sampling partition with a time-varying delay

The sampling time instants associated to the unique sampling partition and the involved specialize as follows

$$t_i^{k_i} = t_k,\ \tau_i(t_k) = \tau(t_k),\ k \geq 0,\ i = 1, \ldots, p.$$

where $\tau(t)$ is bounded as in (2.16).

The expression of the available measurement, $y^a(t)$, involved in system (2.1) specializes as follows

$$y^a(t) = x^{(1)}(t_k - \tau(t_k)),\ \forall t \in [t_k, t_{k+1}[, \tag{2.17}$$

and it is available at the time instant t_k.

Notice that the time instants corresponding to the measurement of the available output, i.e. $t_k - \tau(t_k)$, $k \geq 0$, can be expressed as follows

$$t_k - \tau(t_k) = t - (t - t_k + \tau(t_k)) = t - \bar{\tau}(t), \tag{2.18}$$

where $\bar{\tau}(t)$ is a piecewise continuous function which is defined as

$$\bar{\tau}(t) = t - t_k + \tau(t_k),\ \forall t \in [t_k, t_{k+1}[. \tag{2.19}$$

Notice that since the sampling partition diameter associated to the sampling instants $\{t_k\}_{k\geq 0}$ is bounded by τ_M and from the fact that the original output delay is bounded by τ_M, the delay output $\bar{\tau}(t)$ is bounded and one has

$$\forall t \geq t_0,\ \bar{\tau}(t) = t - t_k + \tau(t_k) \leq \bar{\tau}_M \triangleq \Delta_M + \tau_M. \tag{2.20}$$

The equations of the candidate high gain observer to provide an estimate of the actual state can be written under the following form

$$\left\{ \begin{array}{rcl} \text{For } t & \geq & t_0 : \\ \dot{\hat{x}}(t) & = & A_s(u(t), \hat{x}(t))\hat{x}(t) + \varphi_s(u(t), \hat{x}(t)) - \Gamma(t) \\ \text{For } t & \in & [t^k, t^{k+1}[: \\ \Gamma(t) & = & \theta\Delta_\theta^{-1}K(t)\left(C\hat{x}(t_k - \tau(t_k)) - y^a(t) + g(t) - g(t_k - \tau(t_k))\right) \\ \dot{g}(t) & = & -\theta CK(t)\left(C\hat{x}(t_k - \tau(t_k)) - y^a(t) + g(t) - g(t_k - \tau(t_k))\right), \end{array} \right. \tag{2.21}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the state estimate, $u(t)$ is the system input, $y^a(t)$ denotes the available system output as given by (2.17), $\theta > 0$ is a design parameter, Δ_θ is a $n \times n$ diagonal matrix defined as follows

$$\Delta_\theta = diag\left[I_{n_1},\ \frac{1}{\theta}I_{n_2},\ \dots\ \frac{1}{\theta^{q-1}}I_{n_q}\right], \tag{2.22}$$

and the matrix $K(t) \in \mathbb{R}^{n\times p}$ is issued from the resolution of the following Lyapunov ODE

$$\dot{S}(t) = -\theta\left(S(t) + A_s\left(u(t), \hat{x}(t)\right)^T S(t) + S(t)A_s\left(u(t), \hat{x}(t)\right) - C^T C\right), \tag{2.23}$$

with $S(t_0) = S^T(t_0)$ and where A_s and C are given by (2.13) and (2.3), respectively. The matrix K is expressed as

$$K(t) = S^{-1}(t)C^T \triangleq \begin{pmatrix} K^1(t) \\ \vdots \\ K^q(t) \end{pmatrix} \text{ with } CK(t) = K^1(t) \in \mathbb{R}^{p\times p}. \tag{2.24}$$

Notice that the function $g(t) \in \mathbb{R}^p$ involved in the equations of observer (2.21) can be initialized as follows

$$g(s) = \varphi(s)\ \forall s \in \left[t_0 - \Delta_M - \tau_M,\ t_0\right], \tag{2.25}$$

where $\varphi(s) \in \mathbb{R}^p$ is a arbitrary continuous function over the interval $[t_0 - \Delta_M - \tau_M,\ t_0]$.

Recall that system (2.12) is not uniformly observable, i.e. its observability depends on the input and on its state. Thus, in order to ensure its observability over arbitrarily small time horizons, one shall formulate as in Hernández-González et al. (2016) the following specific excitation condition under the form of an Assumption,

Assumption 2.3.1 *The input sequence $\{u(t)\}_{t\geq t_o}$ is such that for any trajectory $\hat{x}(t)$ of the system (2.21) starting from the compact set X given in Assumption 2.2.1, $\exists T^* > 0$, $\exists\theta^* > 0$, $\exists\delta^* > 0$ such that $\forall\theta \geq \theta^*$ the following persistent excitation condition holds $\forall t \geq \frac{1}{\theta}T^*$,*

$$\int_{t-\frac{1}{\theta}T^*}^{t} \left(C\Phi_{u(t),\hat{x}(t)}(s,t)\right)^T C\Phi_{u(t),\hat{x}(t)}(s,t)ds\ \geq\ \frac{\delta^*}{\theta\alpha(\theta)}\Delta_\theta^2 \tag{2.26}$$

where $\theta > 1$ is a real number, Δ_θ is the diagonal matrix given by (2.22), α is a real-valued positive function satisfying the following properties

$$\alpha(\theta) \geq 1\ \ and\ \ \lim_{\theta\to\infty}\frac{\alpha(\theta)}{\theta^2} = 0 \tag{2.27}$$

and $\Phi_{u(t),\hat{x}(t)}(t,s) \in I\!R^{n\times n}$ *is the transition matrix of the following state affine system*

$$\dot{\xi}(t) = A_s(u(t), \hat{x}(t))\xi(t) \tag{2.28}$$

with state $\xi \in I\!R^n$ *and inputs* $u(t)$ *and* $\hat{x}(t)$ *which are respectively the input of system (2.12) and the state of system (2.21).*

It is worth recalling that the transition matrix is defined as

$$\left\{\begin{array}{rcl} \frac{d}{dt}\left(\Phi_{u(t),\hat{x}(t)}(t,s)\right) & = & A_s(u(t),\hat{x}(t))\Phi_{u(t),\hat{x}(t)}(t,s)\ \forall t \geq s \geq 0 \\ \Phi_{u(t),\hat{x}(t)}(t,t) & = & I_n\ \forall t \geq 0. \end{array}\right. \tag{2.29}$$

The above observer (2.21) actually performs a precise estimation of the system state as shown by the following result.

Theorem 2.3.1 *Consider the system (2.12) with a single partition and a single (time-varying) output delay subject to Assumptions 2.2.1-2.3.1, together with system (2.21). If the upper bound* $\bar{\tau}_M = \Delta_M + \tau_M$ *is such that*

$$\bar{\tau}_M < \frac{\mu_\theta}{\nu_\theta}, \tag{2.30}$$

then, the observation error satisfies the following property

$$\begin{array}{l} \exists \theta_0 >; \forall \theta \geq \theta_0; \forall t \geq \dfrac{T^\star}{\theta};\ \text{one has} \\ \|\tilde{x}^{(k)}(t)\| \leq \rho_\theta(\bar{\tau}_M)e^{-\eta_\theta(\bar{\tau}_M)(t-t_0)} \displaystyle\max_{s\in[t_0-\bar{\tau}_M,t_0]} \|\tilde{x}(s)\|, \end{array} \tag{2.31}$$

where

$$\begin{array}{rcl} \mu_\theta & = & \theta\left(\frac{1}{2} - \sqrt{\frac{\alpha(\theta)}{\theta^2}}\sqrt{\frac{n\bar{\lambda}_M(S)}{\delta^* e^{-T^*}}}\left(L_A x_M + L_\varphi\right)\right), \\ \nu_\theta & = & \theta\dfrac{\alpha(\theta)e^{T^*}}{\delta^*}\ \left(\theta A_M + \sqrt{n}(L_A x_M + L_\varphi)\right), \\ \eta_\theta(\bar{\tau}_M) & = & \left(\mu_\theta - \nu_\theta\bar{\tau}_M\right)e^{-\mu_\theta\bar{\tau}_M}, \\ \rho_\theta(\bar{\tau}_M) & = & \theta^{k-1}\sqrt{\frac{\bar{\lambda}_M\alpha(\theta)e^{T^\star}}{\delta^\star}}\left(1 + \bar{\tau}_M\left(\mu_\theta - \eta_\theta(\bar{\tau}_M)\right)\right), \end{array} \tag{2.32}$$

where x_M *and* A_M *are the respective bounds of the state* x *and the matrix* A *as given in Assumption 2.2.1,* L_A *and* L_φ *are the Lipschitz constants given in Assumption 2.2.2,* δ^*, T^* *and* $\alpha(\theta)$ *are provided by Assumption 2.3.1.*

2.3.1 Proof of Theorem 2.3.1

Let $\tilde{x}(t) \triangleq \hat{x}(t) - x(t)$ be the state observation error associated to system (2.12) and observer (2.21). One gets,

$$\begin{array}{rcl} \dot{\tilde{x}}(t) & = & A_s(u(t),\hat{x}(t))\hat{x}(t) + \varphi_s(u(t),\hat{x}(t)) - \Gamma(t) - \left(A_s\left(u(t),x(t)\right)x(t) + \varphi_s\left(u(t),x(t)\right)\right) \\ \dot{\tilde{x}}(t) & = & A_s(u(t),\hat{x}(t))\tilde{x}(t) + \tilde{A}_s(u(t),\hat{x}(t),x(t))x(t) + \tilde{\varphi}_s(u(t),\hat{x}(t),x(t)) \\ & & -\theta\Delta_\theta^{-1}K(t)(C\tilde{x}(t_k-\tau(t_k)) + g(t) - g(t_k-\tau(t_k)) \end{array} \tag{2.33}$$

where $A_s\left(u(t),\hat{x}(t),x(t)\right)$ and $\tilde{\varphi}_s\left(u(t),\hat{x}(t),x(t)\right)$ are given by

$$\tilde{A}_s\left(u(t),\hat{x}(t),x(t)\right) \triangleq A_s\left(u(t),\hat{x}(t)\right) - A_s\left(u(t),x(t)\right), \tag{2.34}$$

$$\tilde{\varphi}_s\left(u(t),\hat{x}(t),x(t)\right) \triangleq \varphi_s\left(u(t),\hat{x}(t)\right) - \varphi_s\left(u(t),x(t)\right). \tag{2.35}$$

Now, set $\bar{x}(t) = \theta^{q-1}\Delta_\theta\tilde{x}(t)$ as the exponential error, $\bar{g}(t) = \theta^{q-1}g(t)$ and considering the following properties

$$\Delta_\theta A_s(u,\cdot)\Delta_\theta^{-1} = \theta A_s(u,\cdot) \text{ and } C\Delta_\theta^{-1} = C. \tag{2.36}$$

Using (2.36) and considering (2.21), (2.33), one gets

$$\left\{\begin{array}{rcl} \dot{\bar{x}}(t) & = & \theta A_s(u(t),\hat{x}(t))\bar{x}(t) + \theta^{q-1}\Delta_\theta\left(\tilde{A}_s(u(t),\hat{x}(t),x(t))\tilde{x} + \tilde{\varphi}_s(u(t),\hat{x}(t),x(t))\right) \\ & & -\theta K(t)(C\bar{x}(t_k-\tau(t_k)) + \bar{g}(t) - \bar{g}(t_k-\tau(t_k)), \\ \dot{\bar{g}}(t) & = & -\theta K^{(1)}(t)\left(C\bar{x}(t_k-\tau(t_k)) + \bar{g}(t) - \bar{g}(t_k-\tau(t_k))\right) \end{array}\right. \tag{2.37}$$

Adding and subtracting $\theta K(t)C\bar{x}(t) = \theta K(t)\bar{x}^{(1)}(t)$ to the right side of the first equation of the above system (2.37), one gets

$$\begin{aligned} \dot{\bar{x}}(t) =& \theta A_s(u(t),\hat{x}(t))\bar{x}(t) + \theta^{q-1}\Delta_\theta\left(\tilde{A}_s(u(t),\hat{x}(t),x(t))\tilde{x} + \tilde{\varphi}_s(u(t),\hat{x}(t),x(t))\right) \\ & - \theta K(t)(C\bar{x}(t_k-\tau(t_k)) + \bar{g}(t) - \bar{g}(t_k-\tau(t_k)) + \theta K(t)C\bar{x}(t) - \theta K(t)C\bar{x}(t), \\ \dot{\bar{x}}(t) =& \theta\left(A_s(u(t),\hat{x}(t)) - K(t)C\right)\bar{x}(t) + \theta^{q-1}\Delta_\theta\left(\tilde{A}_s(u(t),\hat{x}(t),x(t))\tilde{x} + \tilde{\varphi}_s(u(t),\hat{x}(t),x(t))\right) \\ & + \theta K(t)z(t), \end{aligned} \tag{2.38}$$

and from the fact that $C\bar{x}(t_k - \tau(t_k)) = \bar{x}^{(1)}(t_k - \tau(t_k))$, one can express the auxiliar variable $z(t)$ as follows:

$$z(t) = \left(\bar{x}^{(1)}(t) - \bar{x}^{(1)}(t_k-\tau(t_k))\right) - \left(\bar{g}(t) - \bar{g}(t_k-\tau(t_k))\right). \tag{2.39}$$

Notice that the variable $z(t)$ can be expressed as follows

$$\begin{aligned} z(t) &= \int_{t_k-\tau(t_k)}^{t} C\dot{\bar{x}}(s)ds - \int_{t_k-\tau(t_k)}^{t} \dot{\bar{g}}(s)ds \\ &= \int_{t_k-\tau(t_k)}^{t} \dot{\bar{x}}^{(1)}(s)ds - \int_{t_k-\tau(t_k)}^{t} \dot{\bar{g}}(s)ds \\ &= \int_{t_k-\tau(t_k)}^{t} \left(\dot{\bar{x}}^{(1)}(s) - \dot{\bar{g}}(s)\right) ds \end{aligned} \tag{2.40}$$

Using the first equation of (2.37), the dynamics of $\bar{x}^{(1)} = C\bar{x}$ can be expressed as follows

$$\begin{aligned} \dot{\bar{x}}^{(1)}(t) = {} & \theta A^{(1)}(u(t), \hat{x}^{(1)}(t))\bar{x}^{(2)}(t) \\ & + \theta^{q-1}\left(\tilde{A}_s^{(1)}(u(t), \hat{x}^{(1)}(t), x^{(1)}(t))x^{(2)}(t) + \tilde{\varphi}_s^{(1)}(u(t), \hat{x}^{(1)}(t), x^{(1)}(t))\right) \\ & - \theta K^{(1)}(t)\left(\bar{x}^{(1)}(t_k - \tau(t_k)) + \bar{g}(t) - \bar{g}(t_k - \tau(t_k))\right), \\ = {} & \theta A^{(1)}(u(t), \hat{x}^{(1)}(t))\bar{x}^{(2)}(t) \\ & + \theta^{q-1}\left(\tilde{A}_s^{(1)}(u(t), \hat{x}^{(1)}(t), x^{(1)}(t))x^{(2)}(t) + \tilde{\varphi}_s^{(1)}(u(t), \hat{x}^{(1)}(t), x^{(1)}(t))\right) + \dot{\bar{g}}(t). \end{aligned}$$

Hence, one gets

$$\begin{aligned} \dot{\bar{x}}^{(1)}(t) - \dot{\bar{g}}(t) = {} & \theta A^{(1)}(u(t), \hat{x}^{(1)}(t))\bar{x}^{(2)}(t) \\ & + \theta^{q-1}\left(\tilde{A}_s^{(1)}(u(t), \hat{x}^{(1)}(t), x^{(1)}(t))x^{(2)}(t) + \tilde{\varphi}_s^{(1)}(u(t), \hat{x}^{(1)}(t), x^{(1)}(t))\right) \end{aligned}$$

Integrating from $t_k - \tau(t_k)$ to t both sides of the above equation while bearing in mind the expression of $z(t)$ given by (2.65) leads to

$$\begin{aligned} z(t) = \int_{t_k-\tau(t_k)}^{t} & \left(\theta A^{(1)}(u(s), \hat{x}^{(1)}(s))\bar{x}^{(2)}(s)\right. \\ & \left. + \theta^{q-1}\left(\tilde{A}_s^{(1)}(u(s), \hat{x}^{(1)}(s), x^{(1)}(s))x^{(2)}(s) + \tilde{\varphi}_s^{(1)}(u(s), \hat{x}^{(1)}(s), x^{(1)}(s))\right)\right) ds. \end{aligned} \tag{2.41}$$

In order to verify the stability of the system, let us propose the following candidate Lyapunov function:

$$V(\bar{x}(t)) = \bar{x}^T(t)S(t)\bar{x}(t) \tag{2.42}$$

where $S(t)$ is the matrix defined as in (2.23). Combining (2.38) and (2.23), one gets

$$\begin{aligned} \dot{V}(\bar{x}(t)) \leq {} & -\theta V(\bar{x}(t)) + 2\bar{x}^T\theta S(t)z(t) \\ & + 2\bar{x}^T(t)S(t)\theta^{q-1}\Delta_\theta\left(\tilde{A}\left(u(t), \hat{x}(t), x(t)\right)x(t) + \tilde{\varphi}\left(u(t), \hat{x}(t), x(t)\right)\right) \end{aligned} \tag{2.43}$$

Using the Lipschitz Assumption 2.2.2, one can show that

$$\theta^{q-1}\Delta_\theta\left(\tilde{A}_s(u(t), \hat{x}(t), x(t))x(t) + \tilde{\varphi}_s(u(t), \hat{x}(t), x(t))\right) \leq \sqrt{n}(L_A x_M + L_\varphi)\bar{x}(t) \tag{2.44}$$

Using (2.44), inequality (2.43) leads to

$$\dot{V}\,(\bar{x}(t)) \leq -\,\theta V(\bar{x}(t)) + 2\theta\sqrt{\bar{\lambda}_M}\sqrt{V(\bar{x})}\|z(t)\| + 2\frac{\sqrt{\bar{\lambda}_M}}{\sqrt{\bar{\lambda}_m}}\sqrt{n}(L_A x_M + L_\varphi)V(\bar{x}(t)), \tag{2.45}$$

where $\bar{\lambda}_M$ (resp. $\bar{\lambda}_m$) is the largest (resp. smallest) eigenvalue of $S(t)$.

Similarly, combining (2.44) and (2.41), one can show that

$$\|z(t)\| \leq \left(\theta A_M + \sqrt{n}(L_A x_M + L_\varphi)\right) \int_{t_k-\tau(t_k)}^{t} \sqrt{\frac{V(\bar{x}(s))}{\bar{\lambda}_m}} ds \tag{2.46}$$

Combining (2.45) and (2.46) results in

$$\begin{aligned}\dot{V}(\bar{x}(t)) \leq &-\left(\theta - 2\frac{\sqrt{\bar{\lambda}_M}}{\sqrt{\bar{\lambda}_m}}\sqrt{n}(L_A x_M + L_\varphi)\right) V(\bar{x}(t)) \\ &+ 2\theta\frac{(\theta A_M + \sqrt{n}(L_A x_M + L_\varphi))}{\sqrt{\bar{\lambda}_m}}\sqrt{V(\bar{x}(s))}\int_{t_k-\tau(t_k)}^{t} \sqrt{V(\bar{x}(s))} ds,\end{aligned} \tag{2.47}$$

or equivalently

$$\begin{aligned}\frac{d}{dt}\sqrt{V(\bar{x}(t))} \leq &-\theta\left(\frac{1}{2} - \frac{\sqrt{\bar{\lambda}_M}}{\theta\sqrt{\bar{\lambda}_m}}\sqrt{n}(L_A x_M + L_\varphi)\right)\sqrt{V(\bar{x})} \\ &+ \theta\frac{(\theta A_M + \sqrt{n}(L_A x_M + L_\varphi))}{\sqrt{\bar{\lambda}_m}}\int_{t_k-\tau(t_k)}^{t} \sqrt{V(\bar{x}(s))} ds.\end{aligned}$$

Using the bound of $\bar{\lambda}_m \geq \frac{\delta^\star e^{-T^\star}}{\alpha(\theta)}$, the last inequality results in

$$\begin{aligned}\frac{d}{dt}\sqrt{V(\bar{x}(t))} \leq &-\theta\left(\frac{1}{2} - \sqrt{\frac{\alpha(\theta)}{\theta^2}}\sqrt{\frac{n\bar{\lambda}_M e^{T^\star}}{\delta^\star}}(L_A x_M + L_\varphi)\right)\sqrt{V(\bar{x}(t))} \\ &+ \theta\frac{\alpha(\theta)e^{T^\star}}{\delta^\star}\left(\theta A_M + \sqrt{n}(L_A x_M + L_\varphi)\right)\int_{t-\bar{\tau}_M}^{t} \sqrt{V(\bar{x}(s))} ds, \\ &\triangleq -\mu_\theta\sqrt{V(\bar{x}(t))} + \nu_\theta\int_{t-\bar{\tau}_M}^{t} \sqrt{V(\bar{x}(s))} ds,\end{aligned} \tag{2.48}$$

where the expressions of μ_θ and ν_θ are given by (2.32).

Using (2.18) and (2.20), inequality (2.49) leads to

$$\frac{d}{dt}\sqrt{V(\bar{x}(t))} \leq -\mu_\theta\sqrt{V(\bar{x}(t))} + \nu_\theta\int_{t-\bar{\tau}_M}^{t} \sqrt{V(\bar{x}(s))} ds. \tag{2.49}$$

Since $\bar{\tau}_M \le \frac{\nu_\theta}{\mu_\theta}$, the conditions of lemma B.1.1 are satisfied and one gets

$$\sqrt{V(\bar{x}(t))} \le \left(1 + \bar{\tau}_M \left(\mu_\theta - \eta_\theta(\bar{\tau}_M)\right)\right) e^{-\eta_\theta(\bar{\tau}_M)(t-t_0)} \max_{\nu \in [t_0 - \bar{\tau}_M, t_0]} \sqrt{V(\bar{x}(\nu))}, \tag{2.50}$$

with

$$\eta_\theta(\bar{\tau}_M) = \left(\mu_\theta - \nu_\theta \bar{\tau}_M\right) e^{-\mu_\theta \bar{\tau}_M}. \tag{2.51}$$

Coming back to $\bar{x}(t)$, one gets

$$\begin{aligned}
\|\bar{x}(t))\| &\le \sqrt{\frac{\bar{\lambda}_M}{\bar{\lambda}_m}} \left(1 + \bar{\tau}_M \left(\mu_\theta - \eta_\theta(\bar{\tau}_M)\right)\right) e^{-\eta_\theta(\bar{\tau}_M)(t-t_0)} \max_{\nu \in [t_0 - \bar{\tau}_M, t_0]} \|\bar{x}(\nu)\| \\
&\le \sqrt{\frac{\bar{\lambda}_M \alpha(\theta) e^{T^\star}}{\delta^\star}} \left(1 + \bar{\tau}_M \left(\mu_\theta - \eta_\theta(\bar{\tau}_M)\right)\right) e^{-\eta_\theta(\bar{\tau}_M)(t-t_0)} \max_{\nu \in [t_0 - \bar{\tau}_M, t_0]} \|\bar{x}(\nu)\|. \qquad (2.52)
\end{aligned}$$

From the fact $\bar{x}_i = \theta^{q-i}\tilde{x}_i, i = 1, \ldots, q$, one gets for $k = 1, \ldots, q$,

$$\|\tilde{x}^{(k)}(t))\| \le \rho_\theta(\tau_M) e^{-\eta_\theta(\bar{\tau}_M)(t-t_0)} \max_{\nu \in [t_0 - \bar{\tau}_M, t_0]} \|\tilde{x}(\nu)\|,$$

where ρ_θ is given as in (2.32). The proof of Theorem 2.3.1 is ended. ■

It is demonstrated that the error converges exponentially to zero. In other words, it is shown that the state estimates by the observer, even in the presence of sampling and delay, converge to the trajectories of the original system.

2.4 Multi-rate sampling with multiple time-varying delays

This section focuses on the observer synthesis considering the case of multi-rate sampling output measurements under multiple short time-varying delays bearing in mind the high gain observer provided above. Each of the p components of the output $x^{(1)}(t)$ is available at the time instant $t_i^{k_i}$ with a delay $\tau_i(t_i^{k_i})$. Hence, the underlying available measurement y_i^a can be written in the following form

$$\begin{aligned}
&y_i^a(t) = x_i^{(1)}(t - \bar{\tau}_i(t)) \text{ with} \\
&\bar{\tau}_i(t) = t - t_i^{k_i} + \tau_i(t_i^{k_i}), \ \forall t \in [t_i^{k_i}, t_i^{k_i+1}[,
\end{aligned} \tag{2.53}$$

and it is available at the time instant $t_i^{k_i}$. The overall vector $y^a(t)$ of the available outputs can then be expressed as

$$y^a(t) = \begin{pmatrix} x_1^{(1)}(t - \bar{\tau}_1(t)) \\ \vdots \\ x_p^{(1)}(t - \bar{\tau}_p(t)) \end{pmatrix}, \tag{2.54}$$

where the $\bar{\tau}_i$'s, $i = 1, \ldots, p$ are defined as in (2.53).

Let us associate to each scalar output $y_i^a, i = 1, \ldots, p$, a scalar function g_i the design of which is explained in what follows. Indeed, let us consider the following vector of functions

$$g(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_p(t) \end{pmatrix},$$

and its delayed form

$$g_{\bar{\tau}}(t) = \begin{pmatrix} g_1(t - \bar{\tau}_1(t)) \\ \vdots \\ g_p(t - \bar{\tau}_p(t)) \end{pmatrix}.$$

In the same way, one defines $\hat{x}_{\bar{\tau}}^{(1)}(t)$ as follows:

$$\hat{x}_{\bar{\tau}}^{(1)}(t) = \begin{pmatrix} \hat{x}^{(1)}(t - \bar{\tau}_1(t)) \\ \vdots \\ \hat{x}_p^{(1)}(t - \bar{\tau}_p(t)) \end{pmatrix}$$

where $\hat{x}^{(1)}(t) \in \mathbb{R}^p$ is an appropriate estimate of the actual state system provided by the observer that shall be proposed later. The determination of the function g is achieved through the resolution of the following Delay Differential Equation (DDE)

$$\dot{g}(t) = -\theta K^{(1)}(t) \left(\hat{x}_{\bar{\tau}}^{(1)}(t) - y_i^a(t) + g(t) - g_{\bar{\tau}}(t) \right), \tag{2.55}$$

where $K^{(1)}(t) = CK(t)$ is as defined by (2.24) and $y^a(t)$ as in (2.54). Notice that the initial conditions for the DDE (2.55) is similar to that given by (2.25).

It should be emphasized that the DDE (2.55) is similar to that of $g(t)$ involved in the observer proposed in the previous section, i.e. observer (2.21). However, there is a main difference that has to be put forward and it consists in the updating of the underlying right sides of the corresponding DDE. Indeed, the right member of the DDE involved in (2.21) is updated at the time instants $t^k, k \geq 0$ associated to the single sampling partition. However, the updating the second member of the DDE (2.55) has to be achieved at each time instant where a new delayed measurement becomes available. Hence, let us denote by $t_i^{k_i}$ the last time instant at which the updating of the right side of the DDE (2.55) has been carried. Then, the next time instant for updating this member, that shall be denoted $\kappa_i^{k_i}$, can be expressed as follows

$$\kappa_i^{k_i} = \min_{1 \leq j \leq p, l \geq 0} \{t_j^{l_j}; t_j^{l_j} > t_i^{k_i}\}. \tag{2.56}$$

Notice that the time period between two successive time instant for updating the second member of (2.55), i.e. $\kappa_i^{k_i} - t_i^{k_i}$ is bounded by Δ_M since each sampling partition diameter is so. Indeed, one has

$$\kappa_i^{k_i} - t_i^{k_i} \leq t_i^{k_i+1} - t_i^{k_i} \leq \Delta_M. \tag{2.57}$$

Hence, one gets

$$\forall t \in [t_i^{k_i}, \kappa_i^{k_i}[, t - t_i^{k_i} + \tau_i(t_i^{k_i}) \leq \bar{\tau}_M \triangleq \tau_M + \Delta_M. \tag{2.58}$$

According to the above developments, an admissible system state estimation can be performed by the following high gain observer,

$$\mathcal{OBS}: \begin{cases} \text{For } t & \geq & t_0,\ i = 1, \ldots, p: \\ \dot{\hat{x}}(t) & = & A_s(u(t), \hat{x}(t))\hat{x}(t) + \varphi_s(u(t), \hat{x}(t)) - \Gamma(t) \\ \text{For } t & \in & [t_i^{k_i}, \kappa_i^{k_i}[: \\ \Gamma(t) & = & \theta\Delta_\theta^{-1}K(t)\left(\hat{x}_{\bar{\tau}}^{(1)}(t) - y^a(t) + g(t) - g_{\bar{\tau}}(t)\right) \\ \dot{g}(t) & = & -\theta K^{(1)}(t)\left(\hat{x}_{\bar{\tau}}^{(1)}(t) - y^a(t) + g(t) - g_{\bar{\tau}}(t)\right), \end{cases} \tag{2.59}$$

where the time instants κ_i^k's, $i = 1, \ldots, p$ and $k \geq 0$ are defined as in (2.57). The matrix $K(t)$ is as in (2.24). The initial condition of the above observer (2.59) have to be chosen such that $\hat{x}(0) \in X$ and $g(s) = \psi(s), \forall s \in [t_0 - \Delta_M - \tau_M, t_0]$ where $\psi(t)$ is any arbitrary continuous function over the interval $[t_0 - \Delta_M - \tau_M, t_0]$.

As one would have guessed, the equations of the above observer (2.59) are very similar to those proposed in the case of a simple sampling partition with a single output delay, i.e. (2.21). The unique difference lies in the time intervals on which the dynamics of $g(t)$ is updated: these intervals are of the form $[t^k, t^{k+1}[$ for the latter while they are equal to $[t_i^{k_i}, \kappa_i^{k_i}[$ for the former. Recall that one has $\forall \in [t^k, t^{k+1}[$, the time period $t - t_k + \tau(t_k)$ is bounded as follows $t - t_k + \tau(t_k) \leq \Delta_M + \tau_M$ according to (2.20) for observer (2.21) and the counterpart of this property is given by (2.58). As a result, the observation error associated (2.59) satisfies the same properties as that for (2.59) and this is summarized in the following Theorem.

Theorem 2.4.1 *Consider system (2.12) with multi-rate sampling and multiple (time-varying) short output delays subject to Assumptions 2.2.1-2.3.1, together with system (2.59). If the upper bound $\bar{\tau}_M = \Delta_M + \tau_M$ is such that*

$$\bar{\tau}_M < \frac{\mu_\theta}{p\nu_\theta}, \tag{2.60}$$

where μ_θ and ν_θ are as in (2.32), then each component $\tilde{x}^{(k)}$ of the observation error $\tilde{x}(t) = \hat{x}(t) - x(t)$ satisfies the following property

$$\begin{aligned} &\exists \theta_0 >; \forall \theta \geq \theta_0; \forall t \geq \frac{T^\star}{\theta}; \text{ one has} \\ &\|\tilde{x}^{(k)}(t)\| \leq \bar{\rho}_\theta(\bar{\tau}_M) e^{-\bar{\eta}_\theta(\bar{\tau}_M)(t-t_0)} \max_{s\in[t_0-\bar{\tau}_M, t_0]} \|\tilde{x}(s)\|, \end{aligned} \tag{2.61}$$

where

$$\begin{aligned} \bar{\rho}_\theta(\bar{\tau}_M) &= \theta^{k-1}\sqrt{\frac{\bar{\lambda}_M \alpha(\theta) e^{T^\star}}{\delta^\star}} \left(1 + \bar{\tau}_M \left(\mu_\theta - \bar{\eta}_\theta(\bar{\tau}_M)\right)\right), \\ \bar{\eta}_\theta(\bar{\tau}_M) &= \left(\mu_\theta - p\nu_\theta \bar{\tau}_M\right) e^{-\mu_\theta \bar{\tau}_M}, \end{aligned} \tag{2.62}$$

and all the other variables keep the same meaning as in Theorem 2.3.1.

2.4.1 Proof of Theorem 2.4.1

Let us consider $\tilde{x}(t) = \hat{x}(t) - x(t)$ as the observation error associated to system (2.12) and observer (2.59) and set $\bar{g}(t) = \theta^{q-1} g(t)$ and $\bar{x}(t) = \theta^{q-1}\Delta_\theta \tilde{x}(t)$ where Δ_θ is given by (2.22), one gets

$$\left\{ \begin{aligned} \dot{\bar{x}}(t) &= \theta A_s(u(t), \hat{x}(t))\bar{x}(t) + \theta^{q-1}\Delta_\theta \left(\tilde{A}_s(u(t), \hat{x}(t), x(t))\tilde{x} + \tilde{\varphi}_s(u(t), \hat{x}(t), x(t))\right) \\ &\quad -\theta K(t)(\bar{x}^{(1)}_{\bar{\tau}}(t) - y^{(a)} + \bar{g}(t) - \bar{g}_{\bar{\tau}}(t)), \\ \dot{\bar{g}}(t) &= -\theta K^{(1)}(t) \left(\bar{x}^{(1)}_{\bar{\tau}}(t) - y^{(a)} + \bar{g}(t) - \bar{g}_{\bar{\tau}}(t)\right) \end{aligned} \right. \tag{2.63}$$

where $\tilde{A}_s\left(u(t), \hat{x}(t), x(t)\right)$ and $\tilde{\varphi}_s\left(u(t), \hat{x}(t), x(t)\right)$ are given by Eq. (2.34) and (2.35), respectively.

Adding and subtracting $\theta K(t) C\bar{x}(t) = \theta K(t)\bar{x}^{(1)}(t)$ to the right side of the first equation of the above system (2.63) and from the fact that $C\bar{x}_{\bar{\tau}}(t) = \bar{x}^{(1)}_{\bar{\tau}}(t)$, one gets

$$\begin{aligned} \dot{\bar{x}}(t) =&\theta \left(A_s(u, \hat{x}) - K(t)C\right)\bar{x}(t) + \theta^{q-1}\Delta_\theta \left(\tilde{A}_s(u(t), \hat{x}(t), x(t))\tilde{x} + \tilde{\varphi}_s(u(t), \hat{x}(t), x(t))\right) \\ &+ \theta K(t) z(t), \end{aligned} \tag{2.64}$$

where

$$z(t) = \bar{x}^{(1)}(t) - \bar{x}^{(1)}_{\bar{\tau}}(t) - (\bar{g}(t) - \bar{g}_{\bar{\tau}}(t)). \tag{2.65}$$

Let $i \in \{1, \ldots, p\}$. From the definitions of $\tilde{x}^{(1)}_{\bar{\tau}}(t)$, $g_{\bar{\tau}}(t)$ (or equivalently $\bar{x}^{(1)}_{\bar{\tau}}(t)$, $\bar{g}_{\bar{\tau}}(t)$) and that of $\bar{\tau}(t)$ given by (2.59), equation (2.65) leads for all $i \in \{1, \ldots, p\}$ to

$$\begin{aligned} z_i(t) &= \bar{x}^{(1)}_i(t) - \bar{x}^{(1)}_i(t_i^{(k_i)} - \tau(t_i^{(k_i)})) - \bar{g}_i(t) - \bar{g}_i(t_i^{(k_i)} - \tau(t_i^{(k_i)})) \\ &= \int_{t_i^{(k_i)} - \tau(t_i^{(k_i)})}^{t} \left(\dot{\bar{x}}^{(1)}_i(s) - \dot{\bar{g}}_i(s)\right) ds, \; \forall t \in [t^{(k_i)}, t_i^{(k_i+1)}[. \end{aligned}$$

Using (2.58), the above equality leads to

$$
\begin{aligned}
|z_i(t)| &= \left| \int_{t-\bar{\tau}(t)}^{t} \dot{\bar{x}}_i^{(1)}(s) - \dot{\bar{g}}_i(s) ds \right| \\
&\leq \int_{t-\bar{\tau}(t)}^{t} \left| \dot{\bar{x}}_i^{(1)}(s) - \dot{\bar{g}}_i(s) \right| ds \\
&\leq \int_{t-\bar{\tau}_M}^{t} \left| \dot{\bar{x}}_i^{(1)}(s) - \dot{\bar{g}}_i(s) \right| ds \\
&\leq \int_{t-\bar{\tau}_M}^{t} \left\| \dot{\bar{x}}^{(1)}(s) - \dot{\bar{g}}(s) \right\| ds. \qquad (2.66)
\end{aligned}
$$

It should be emphasized that the last inequality (2.66) is valid for every $i \in \{1, \ldots, p\}$ and every $t \in [t_i^{(k_i)}, t_i^{(k_i+1)}[$. Hence, one gets

$$
\|z(t)\| \leq \sum_{i=1}^{p} |z_i(t)| \ \leq \ p \int_{t-\bar{\tau}_M}^{t} \|\dot{\bar{x}}^{(1)}(s) - \dot{\bar{g}}(s)\| ds. \qquad (2.67)
$$

As in the proof of Theorem 2.3.1 and by using the first equation of (2.63), the dynamics of $\bar{x}^{(1)}(t) = C\bar{x}(t)$ can be expressed as follows

$$
\begin{aligned}
\dot{\bar{x}}^{(1)}(t) =& \theta A_s^{(1)}(u(t), \hat{x}^{(1)}(t))\bar{x}^{(2)}(t) \\
&+ \theta^{q-1} \left(\tilde{A}_s^{(1)}(u(t), \hat{x}^{(1)}(t), x^{(1)}(t))x^{(2)}(t) + \tilde{\varphi}_s^{(1)}(u(t), \hat{x}^{(1)}(t), x^{(1)}(t)) \right) + \dot{\bar{g}}(t).
\end{aligned}
$$

Hence, one gets

$$
\begin{aligned}
\dot{\bar{x}}^{(1)}(t) - \dot{\bar{g}}(t) =& \theta A_s^{(1)}(u(t), \hat{x}^{(1)}(t))\bar{x}^{(2)}(t) \\
&+ \theta^{q-1} \left(\tilde{A}_s^{(1)}(u(t), \hat{x}^{(1)}(t), x^{(1)}(t))x^{(2)}(t) + \tilde{\varphi}_s^{(1)}(u(t), \hat{x}^{(1)}(t), x^{(1)}(t)) \right).
\end{aligned}
$$

The above inequality leads to

$$
\begin{aligned}
\|\dot{\bar{x}}^{(1)}(t) - \dot{\bar{g}}(t)\| \leq& \theta \|A_s^{(1)}(u(t), \hat{x}^{(1)}(t))\| \|\bar{x}^{(2)}(t)\| \\
&+ \theta^{q-1} \left\| \tilde{A}_s^{(1)}(u(t), \hat{x}^{(1)}(t), x^{(1)}(t))x^{(2)}(t) + \tilde{\varphi}_s^{(1)}(u(t), \hat{x}^{(1)}(t), x^{(1)}(t)) \right\|.
\end{aligned}
\qquad (2.68)
$$

Again, inequality (2.44) derived in the proof of Theorem 2.3.1 is valid. Combining this inequality with the above inequality (2.68) and using the bound of A given by (2.6), leads to

$$
\begin{aligned}
\|\dot{\bar{x}}^{(1)}(t) - \dot{\bar{g}}(t)\| &\leq \theta A_M \|\bar{x}^{(2)}(t)\| + \sqrt{n}(L_A x_M + L_\varphi)\|\bar{x}(t)\| \\
&\leq \left(\theta A_M + \sqrt{n}(L_A x_M + L_\varphi)\right) \|\bar{x}(t)\|. \qquad (2.69)
\end{aligned}
$$

Combining inequalities (2.67) and (2.69) leads to

$$\|z(t)\| \;\leq\; p\left(\theta A_M + \sqrt{n}(L_A x_M + L_\varphi)\right)\int_{t-\bar{\tau}_M}^{t} \|\bar{x}(s)\|ds. \tag{2.70}$$

Now, one proposes the following candidate Lyapunov function

$$V\left(\bar{x}(t)\right) = \bar{x}^T(t)S(t)\bar{x}(t) \tag{2.71}$$

where $S(t)$ is the matrix defined as in (2.23). Proceeding as in the proof of Theorem 2.3.1, i.e. by combining (2.64) and (2.23), one gets

$$\begin{aligned}
\dot{V}\left(\bar{x}(t)\right) \;&\leq\; -\theta V(\bar{x}(t)) + 2\bar{x}^T(t)\theta z(t) \\
&\quad +2\bar{x}^T(t)S(t)\theta^{q-1}\Delta_\theta\left(\tilde{A}\left(u(t),\hat{x}(t),x(t)\right)x + \tilde{\varphi}\left(u(t),\hat{x}(t),x(t)\right)\right), \\
&\leq\; -\theta V(\bar{x}(t)) + 2\theta\frac{\sqrt{V(\bar{x}(t))}}{\sqrt{\lambda_m}}\|z(t)\| \\
&\quad +2\sqrt{\lambda_M}\sqrt{V(\bar{x}(t))}\left(\|\theta^{q-1}\Delta_\theta\left(\tilde{A}\left(u(t),\hat{x}(t),x(t)\right)x(t) + \tilde{\varphi}\left(u(t),\hat{x}(t),x(t)\right)\right)\right\|.
\end{aligned}$$

Again, using (2.44) and substituting in the above inequality $\|z(t)\|$ by its bound given by (2.70) leads to

$$\begin{aligned}
\dot{V}(\bar{x}(t)) \;&\leq\; -\theta V(\bar{x}(t)) + 2\frac{p}{\lambda_m}\theta\left(\theta A_M + \sqrt{n}(L_A x_M + L_\varphi)\right)\sqrt{V(\bar{x}(t))}\int_{t-d_M}^{t}\|\sqrt{V(\bar{x}(s))}ds \\
&\quad +2\sqrt{n}\frac{\sqrt{\lambda_M}}{\sqrt{\lambda_m}}(L_A x_M + L_\varphi)V(\bar{x}(t)) \\
&=\; -\left(\theta - 2\frac{\sqrt{\lambda_M}}{\sqrt{\lambda_m}}\sqrt{n}(L_A x_M + L_\varphi)\right)V(\bar{x}(t)) \\
&\quad +p\frac{2}{\lambda_m}\left(\theta A_M + \sqrt{n}(L_A x_M + L_\varphi)\right)\sqrt{V(\bar{x}(t))}\int_{t-\bar{\tau}_M}^{t}\sqrt{V(\bar{x}(s))}ds.
\end{aligned} \tag{2.72}$$

Notice that the last inequality (2.72) is almost identical to that generated in the single sampling partition/single delay case and given by (2.47). The only difference between these two inequality lies in the fact that the second term on the right side of inequality (2.72) is multiplied by the scalar p.

Hence, proceeding in a similar as in the proof of Theorem 2.3.1 leads to,

$$\frac{d}{dt}\sqrt{V(\bar{x}(t))} \;\leq\; -\mu_\theta\sqrt{V(\bar{x}(t))} + p\nu_\theta\int_{t-\bar{\tau}_M}^{t}\sqrt{V(\bar{x}(s))}ds, \tag{2.73}$$

where μ_θ and ν_θ are as given by (2.32) of Theorem 2.3.1.

Since $\bar{\tau}_M \leq p\dfrac{\nu_\theta}{\mu_\theta}$, conditions of lemma B.1.1 are satisfied and one gets

$$\sqrt{V(\bar{x}(t))} \leq (1 + \bar{\tau}_M (\mu_\theta - \bar{\eta}_\theta(\bar{\tau}_M)))\, e^{-\bar{\eta}_\theta(\bar{\tau}_M)(t-t_0)} \max_{s\in[t_0-\bar{\tau}_M, t_0]} \sqrt{V(\bar{x}(s))}, \tag{2.74}$$

with

$$\bar{\eta}_\theta(\bar{\tau}_M) = (\mu_\theta - p\nu_\theta\bar{\tau}_M)\, e^{-\mu_\theta\bar{\tau}_M}. \tag{2.75}$$

Finally, coming back to $\bar{x}^{(k)}$, one gets for $k = 1, \ldots, q$,

$$\|\tilde{x}^{(k)}(t))\| \leq \bar{\rho}_\theta(\bar{\tau}_M) e^{-\bar{\eta}_\theta(\bar{\tau}_M)(t-t_0)} \max_{s\in[t_0-\bar{\tau}_M, t_0]} \|\tilde{x}(s)\|,$$

where $\bar{\rho}_\theta$ is given as in (2.62). This ends the proof of the Theorem. ■

Thereby, it is clear that even in the presence of multiple sampling and multiple delays, the proposed observer is able to estimate the system states.

2.5 Application to HIV infection system

In order to validate the effectiveness of the observer, different simulations considering different delay and sampling scenarios are presented. The observer is applied to the estimation of variables and parameters of the HIV biological system.

The HIV is a retrovirus that uses the $CD4^+$ T-cells of a healthy person, which are of vital importance in the immune system, to replicate. HIV attacks $CD4^+$ T-cells in the body that help fight disease and other infections. Attacking these cells makes the infected body vulnerable to other infections and diseases. Monitoring the variables and parameters of the model leads physicians to see the effectiveness of the treatment provided to their patients. However, due to different phenomena, there are delays in the model.

Bearing in mind the above, the following model of HIV infection proposed in (Culshaw and Ruan, 2000; Sharafian et al., 2020) is defined as follows:

$$\mathcal{SYS} : \begin{cases} \dot{T}(t) = s - \mu_T T(t) + rT(t)\left(1 - \frac{T(t)+I(t)}{T_{\max}}\right) - k_1 V(t)T(t) \\ \dot{I}(t) = k_1^{'} V(t)T(t) - \mu_I I(t) \\ \dot{V}(t) = N\mu_b I(t) - k_1 V(t)T(t) - \mu_V V(t) \\ y^{(a)}(t) = \begin{bmatrix} T(t - \bar{\tau}_1(t)) \\ I(t - \bar{\tau}_2(t)) \end{bmatrix} \end{cases} \tag{2.76}$$

where T is the population of healthy CD4$^+$ T-cells, I represents the population of infected CD4$^+$ T-cells and V is the free HIV virus particles in the blood of HIV individual.

The parameter values of system (2.76) are shown in Table 2.1. Note that [mm] means millimeters.

Parameter	**Value**	**Unit**
s	10	day^{-1}mm^{-1}
μ_T	0.2	day^{-1}
μ_b	0.24	day^{-1}
μ_I	0.26	day^{-1}
μ_V	2.4	day^{-1}
T_{max}	1500	mm^{-1}
k_1	2.4×10^{-5}	mm^3 day^{-1}
r	0.03	day^{-1}
k_1'	2×10^{-5}	mm^3 day^{-1}
N	1400	-

Table 2.1: Physical parameters of the HIV system

Now, one defines the following state vector $x(t) = [x^{(1)T}\ x^{(2)}]^T \in \mathbb{R}^3$ and it is partitioned as follows

$$x(t) = \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \text{ with } x^{(1)}(t) = \begin{bmatrix} x_1^{(1)}(t) & = & T(t) \\ x_2^{(1)}(t) & = & I(t) \end{bmatrix} \text{ and } x^{(2)}(t) = \quad V(t).$$

By bearing in mind the above, one can rewrite system (2.76) as follows:

$$\mathcal{SYS} : \begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 0 & -k_1 x_1^{(1)}(t) \\ 0 & 0 & k_1' x_1^{(1)}(t) \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} s - \mu_T x_1^{(1)}(t) + r x_1^{(1)}(t)\left(1 - \frac{x_1^{(1)}(t) + x_2^{(1)}(t)}{T_{\max}}\right) \\ -\mu_I x_2^{(1)}(t) \\ N\mu_b x_2^{(1)}(t) - k_1 x_1^{(1)}(t) x_1^{(2)}(t) - \mu_V x_1^{(2)}(t) \end{bmatrix} \\ y^{(a)}(t) \quad = C x_{\bar{\tau}}(t) = x_{\bar{\tau}}^{(1)}(t) \end{cases} \tag{2.77}$$

with $C = \begin{bmatrix} I_2 & 0_{2\times 1} \end{bmatrix}$. Recall that the actual output $y(t)$ is unknown, and the available output is $y^{(a)}(t)$ that is only available in instants of time and is delayed. In other words, the available output is defined as follows:

$$y^{(a)} = x_{\bar{\tau}}^{(1)}(t) = \begin{pmatrix} x_1^{(1)}(t - \bar{\tau}_1(t)) \\ x_2^{(1)}(t - \bar{\tau}_1(t)) \end{pmatrix} \tag{2.78}$$

with

$$\begin{cases} \bar{\tau}_1(t) & = t - t_1^{(k_1)} + \tau_1(t_1^{(k_1)}),\ \forall t \in \left[t_1^{(k_1)}, t_1^{(k_1+1)}\right[, k_1 \geq 0 \\ \bar{\tau}_2(t) & = t - t_2^{(k_2)} + \tau_2(t_2^{(k_2)}),\ \forall t \in \left[t_2^{(k_2)}, t_2^{(k_2+1)}\right[, k_2 \geq 0 \end{cases} \tag{2.79}$$

It is important to note that each of the signals has a maximum sampling delay denoted by $\Delta_i, i = 1, 2$ and a maximum delay value denoted by $\tau_i, i = 1, 2$. The maximum values of sampling (Δ_M) and time delay (τ_M) are selected considering the upper bound of the outputs, i.e.

$$\begin{cases} \Delta_M = & \max\{\ \Delta_1,\quad \Delta_2\ \} \\ \tau_M = & \max\{\ \tau_1,\quad \tau_2\ \} \end{cases} \tag{2.80}$$

The objective is to demonstrate the ability of the observer to reconstruct the states continuously, even in the presence of multiple sampling and delay. For this, one designs an observer of the form (2.59) as follows:

$$\mathcal{OBS}: \begin{cases} \text{For } t & \geq & t_0,\ i = 1, \ldots, p: \\ \dot{\hat{x}}(t) & = & A_s(\hat{x}(t))\hat{x}(t) + \varphi_s(\hat{x}(t)) - \Gamma(t) \\ \text{For } t & \in & [t_i^{k_i}, \kappa_i^{k_i}[: \\ \Gamma(t) & = & \theta\Delta_\theta^{-1}K(t)\left(\hat{x}_{\bar{\tau}}^{(1)}(t) - y^a(t) + g(t) - g_{\bar{\tau}}(t)\right) \\ \dot{g}(t) & = & -\theta K^{(1)}(t)\left(\hat{x}_{\bar{\tau}}^{(1)}(t) - y^a(t) + g(t) - g_{\bar{\tau}}(t)\right), \\ g(s) & = & 0_{2\times 1}, \forall s \in [-\Delta_M - \tau_M, 0], \end{cases} \tag{2.81}$$

with:

$$g(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \in \mathbb{R}^p,\ \ g_{\bar{\tau}}(t) = \begin{pmatrix} g_1(t - \bar{\tau}_1(t)) \\ g_2(t - \bar{\tau}_2(t)) \end{pmatrix} \in \mathbb{R}^p,$$
$$\hat{x}_{\bar{\tau}}^{(1)}(t) = \begin{pmatrix} \hat{x}_1^{(1)}(t - \bar{\tau}_1(t)) \\ \hat{x}_2^{(1)}(t - \bar{\tau}_1(t)) \end{pmatrix}.$$

Notice that all the following simulations that shall be given were obtained without proceeding to the saturation of the nonlinearities and the state of the system is bounded for the considered initial conditions.

Two sets of simulation results are presented and in both cases the initial conditions of the system and observer states are set as follows:

$$x(0) = \begin{bmatrix} 1000 \\ 0 \\ 10 \times 10^{-3} \end{bmatrix}, \hat{x}(s) = \begin{bmatrix} 1200 \\ 200 \\ 50 \times 10^{-3} \end{bmatrix} \text{with } s \in [-\tau_M, 0],$$

and the initial observer parameters are

$$S(0) = I_{3\times3},\ \Delta_\theta = \begin{bmatrix} I_{2\times2} & 0_{2\times1} \\ 0_{1\times2} & \frac{1}{\theta} \end{bmatrix} \text{ and } \theta = 2.$$

The first set of simulation results deals with very short values of the sampling partition diameters and the upper bound of the time-varying output delays. The first output is sampled with a constant sampling period $\Delta_1 = 0.01$ day and exhibits a time-varying delay $\tau_1 = 0.03(0.75 + 0.25\sin(0.4t))$ day. For the second output, the value of the underlying constant sampling period is $\Delta_2 = 0.02$ day and the expression of the associated time-varying delay is $\tau_2(t) = 0.05(0.75 + 0.25\cos(0.8t))$ day. Considering the proposed in (2.80), one obtains the upper limits as follows:

$$\begin{cases} \Delta_M = & \max\{\ \Delta_1,\ \ \Delta_2\ \} = \max\{\ 0.01,\ \ 0.02\ \}\text{ day } = 0.02 \text{ day}, \\ \tau_M = & \max\{\ \tau_1,\ \ \tau_2\ \} = \max\{\ 0.03,\ \ 0.05\ \}\text{ day } = 0.05 \text{ day}. \end{cases}$$

This results in an upper bound of the involved delays is $\bar{\tau}_M = \Delta_M + \tau_M = 0.07$ day. The performance of the time-varying delay in each output is shown in Figure 2.1. The estimates of the unknown actual state variables provided by the observer are compared to their true values, issued from the simulation of system (2.76), in Figures 2.2- 2.4. More precisely, the estimates of the concentration of healthy and infected CD4$^+$ T-cells are compared to their true unknown time evolution in Figures 2.2-2.3, respectively, while Figure 2.4 shows the estimate of the free HIV. Note that the estimation of free HIV is of great interest, since it is an important indicator of the degree of progression of the infection in the patient, the effectiveness of antiretroviral therapy (ART), the risk of transmission to others, detection of viral rebounds, among others.

It should be emphasized that the available outputs of the system are subject to different sampling partitions and that each of these outputs is available with a specific time-varying delay. The obtained results clearly show that the proposed observer allowed us to quickly provide continuous estimates of the overall system actual state from the available sampled delayed outputs. This is clearly shown in Figure 2.5 that shows the observation error

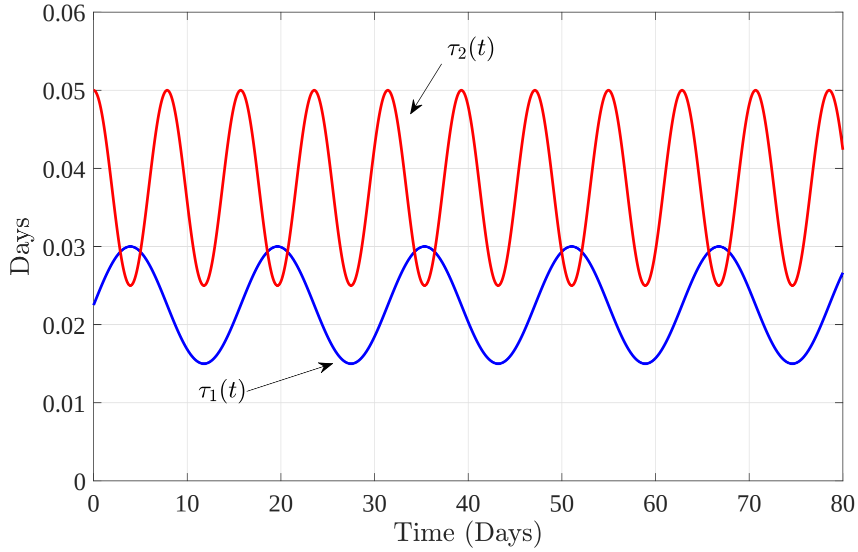

Figure 2.1: First set of simulation: Performance of time-varying delay in each output

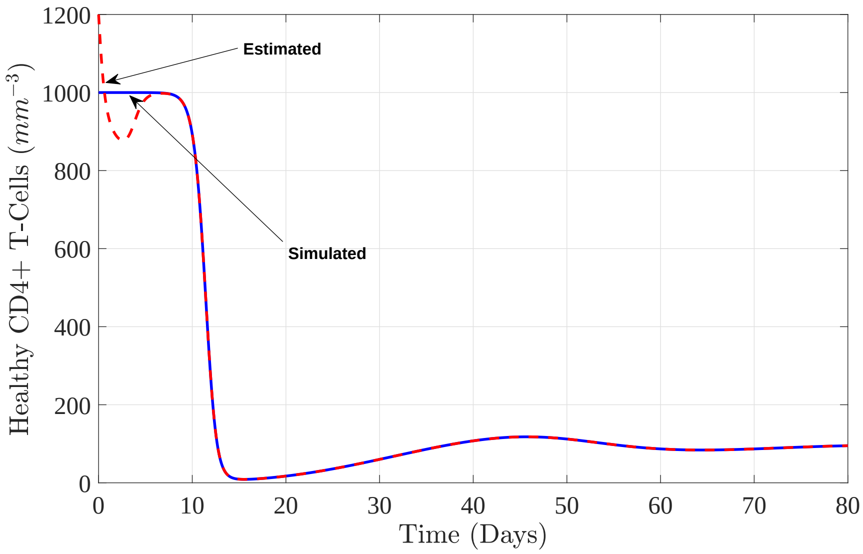

Figure 2.2: First set of simulation: Estimation of the concentration of healthy CD4$^+$ T-cells by the observer

norm. It is clearly shown that it converges to zero, indicating that all states are correctly estimated.

The second set of simulations deals with a more challenging challenge in terms of sampling and delay. The first output is sampled with a constant sampling period $\Delta_1 = 1$ day and exhibits a time-varying delay $\tau_1(t) = 3(0.75 + 0.25 \sin(0.2t))$ day. The second output presents a sampling period equal to $\Delta_2 = 2$ day and the expression of the associated time-varying delay is $\tau_2(t) = 5(0.75 + 0.25 \cos(0.4t))$ day. According to (2.80), the upper

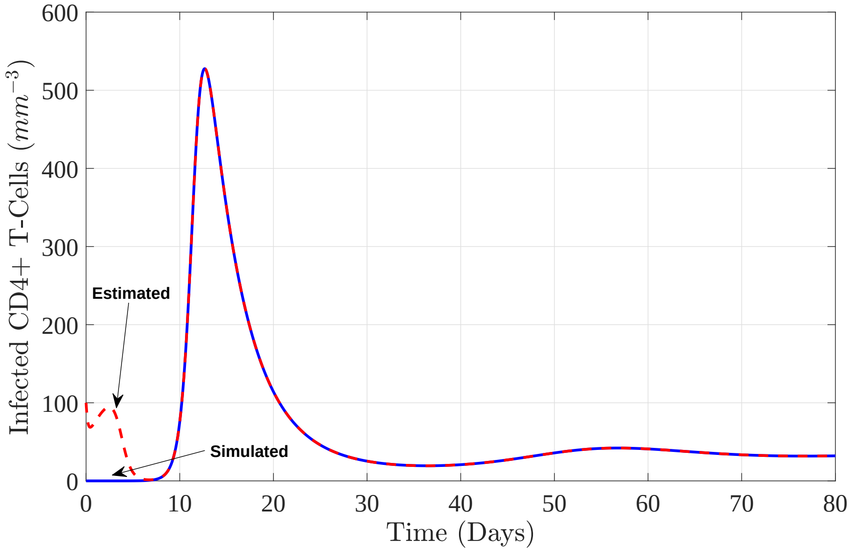

Figure 2.3: First set of simulation: Estimation of the concentration of infected CD4$^+$ T-cells by the observer

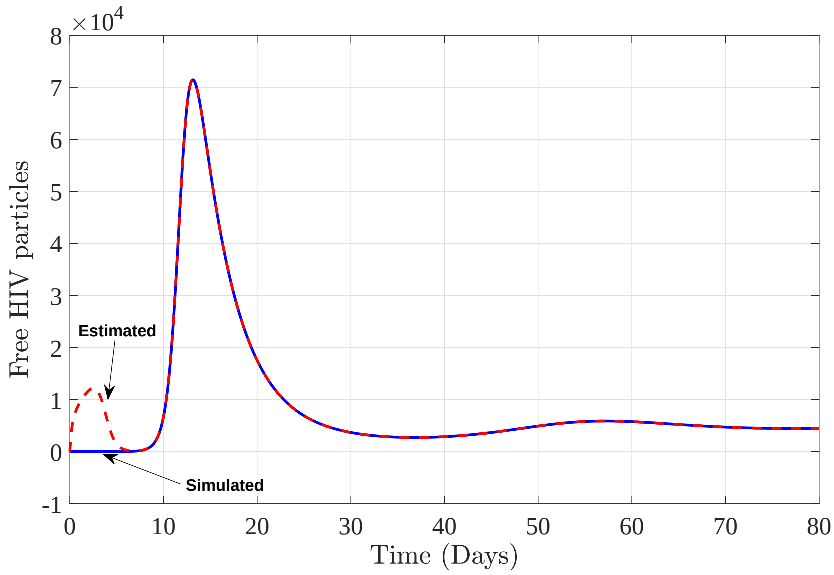

Figure 2.4: First set of simulation: Estimation of the free HIV particles by the observer

limits are as follows:

$$\begin{cases} \Delta_M = & \max\{ \ \Delta_1, \quad \Delta_2 \ \} = \max\{ \ 1, \quad 2 \ \} \text{ day } \ = 2 \text{ day}, \\ \tau_M = & \max\{ \ \tau_1, \quad \tau_2 \ \} = \max\{ \ 3, \quad 5 \ \} \text{ day } \ = 5 \text{ day}. \end{cases}$$

For this simulation, the upper bound of the involved delays is $\bar{\tau}_M = \Delta_M + \tau_M = 7$ day, which is ten times greater than the first set of simulation. The performance of the time-varying delay in each output is shown in Figure 2.6. The estimates of the

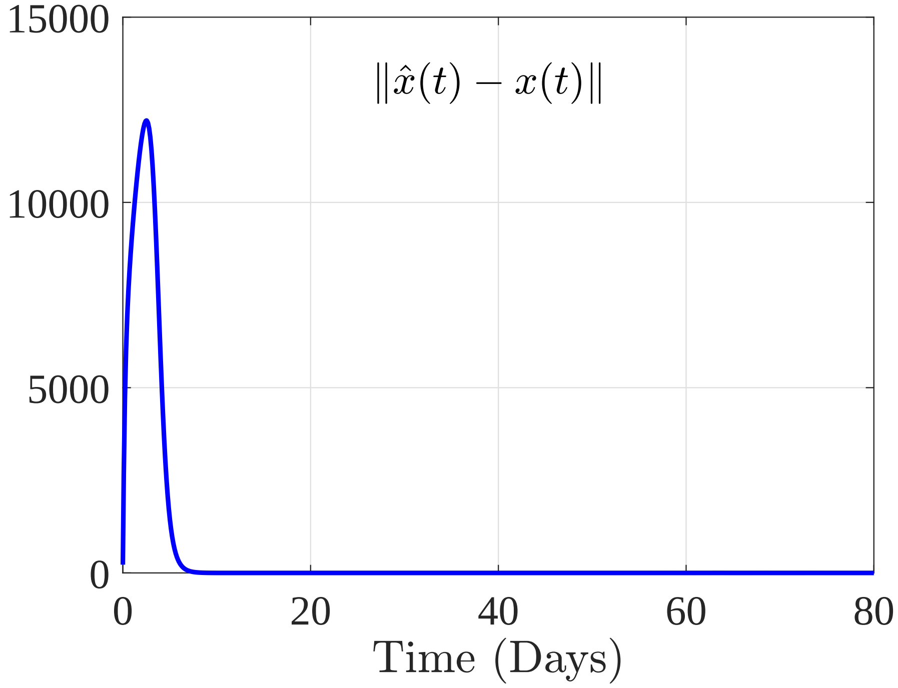

Figure 2.5: First set of simulation: Observation error norm $\|\hat{x}(t) - x(t)\|$

unknown actual state variables provided by the observer are shown in Figures 2.7- 2.9. The estimation of the concentration of healthy and infected CD4$^+$ T-cells are compared to their true unknown time evolution in Figures 2.7-2.8, respectively, while Figure 2.9 shows the estimate of the free HIV. It is noted that even with increasing sampling and delay in the system, the observer manages to correctly estimate the states of the original system. However, it presents higher values in the transient and takes more time to converge, as shown in Figure 2.10.

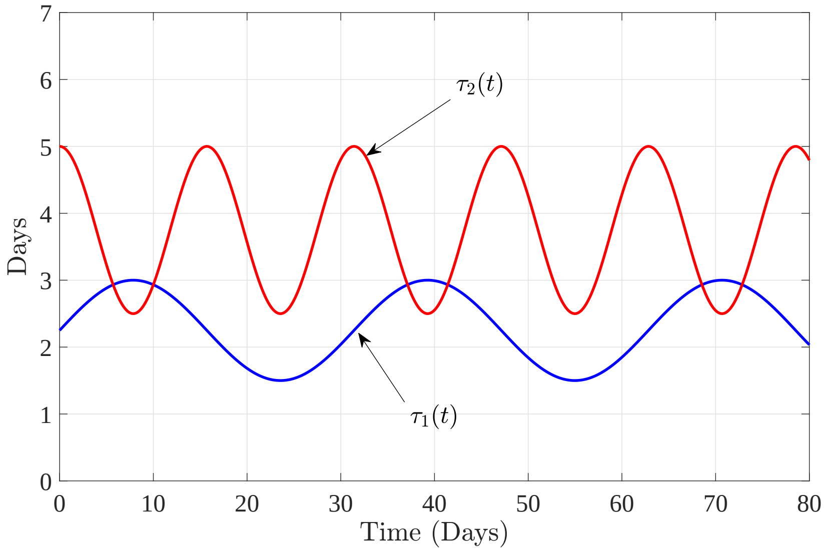

Figure 2.6: Second set of simulation: Performance of time-varying delay in each output

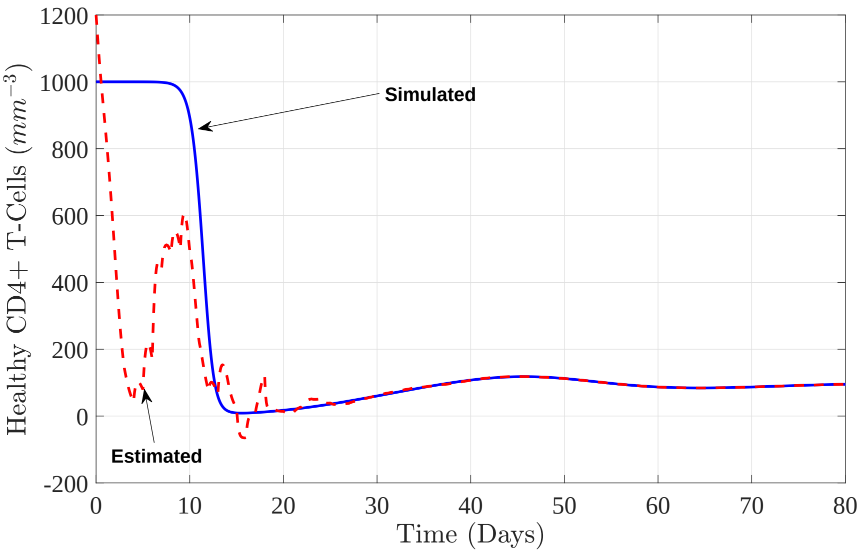

Figure 2.7: Second set of simulation: Estimation of the concentration of healthy CD4$^+$ T-cells by the observer

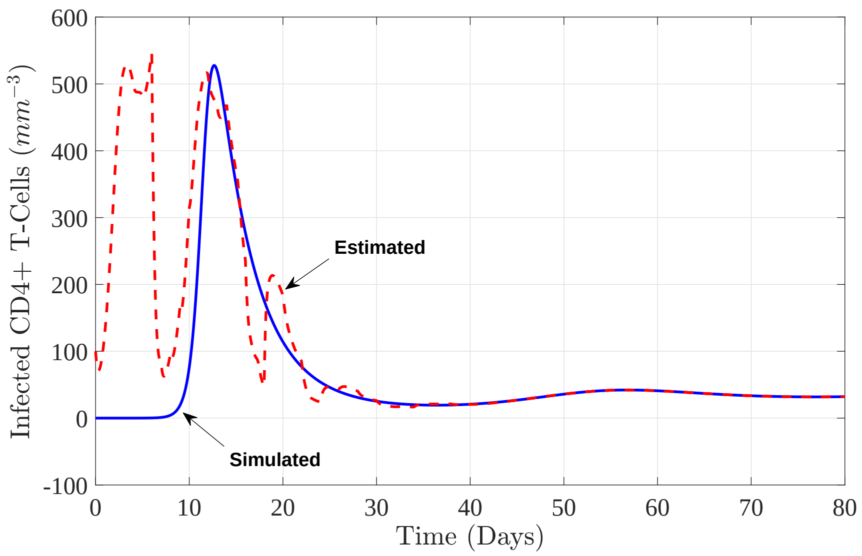

Figure 2.8: Second set of simulation: Estimation of the concentration of infected CD4$^+$ T-cells by the observer

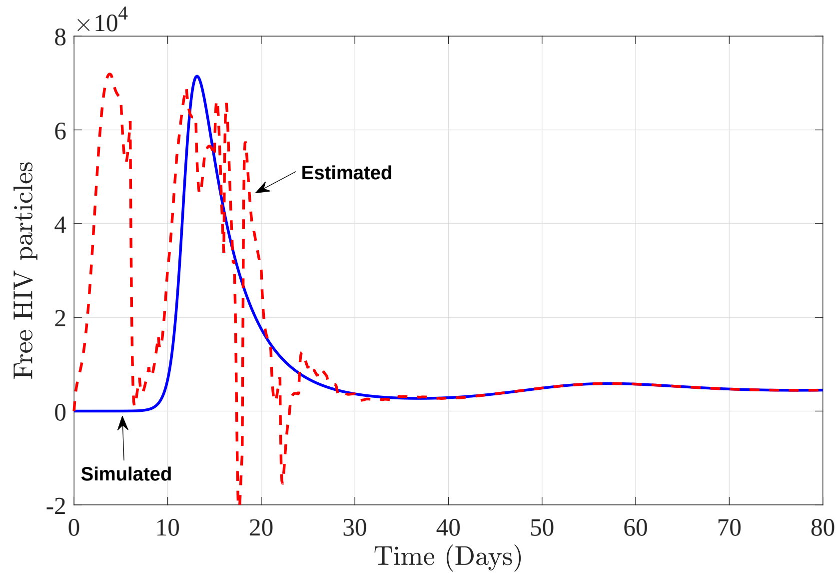

Figure 2.9: Second set of simulation: Estimation of the free HIV particles by the observer

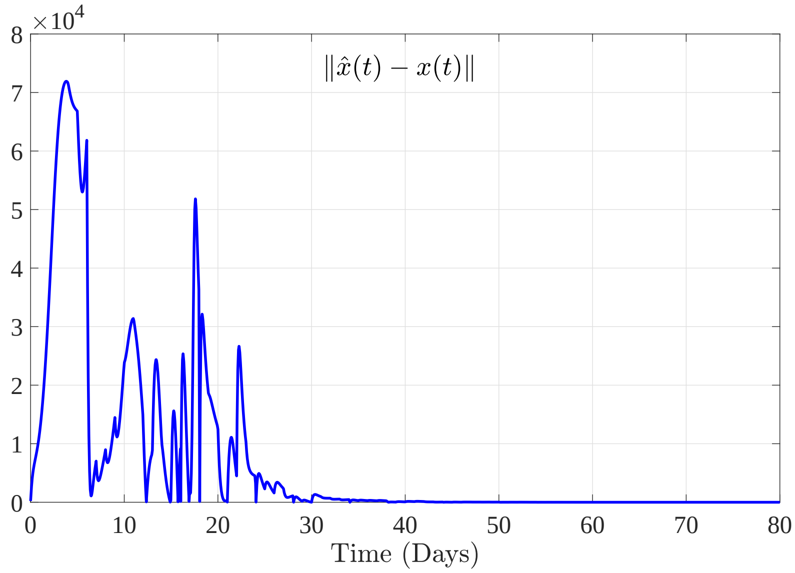

Figure 2.10: Second set of simulation: Observation error norm $\|\hat{x}(t) - x(t)\|$

2.5.1 Estimation of unknown parameters

In this section, the class of the non-uniformly observable system is exploited. For this, the system shown in (2.76) is considered. However, the parameters s and r are considered to be unknown.

Now the challenge is to obtain the continuous estimation of the state variables and the two unknown parameters. One option to solve this may be to propose an adaptive observer; however, an alternative is to propose the following state vector $x(t) = [T(t)\ I(t)\ V(t)\ s\ r]^T$ and considering that s and r are constant, their time derivatives are $\dot{s} = \dot{r} = 0$.

The state vector $x(t)$ is partitioned as follows:

$$x(t) = \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \text{ with } x^{(1)}(t) = \begin{bmatrix} x_1^{(1)}(t) & = & T(t) \\ x_2^{(1)}(t) & = & I(t) \end{bmatrix} \text{ and } x^{(2)}(t) = \begin{bmatrix} x_1^{(2)}(t) & = & V(t) \\ x_2^{(2)}(t) & = & s \\ x_3^{(2)}(t) & = & r \end{bmatrix}.$$

By considering the above, the system (2.76) can be rewritten under the form

$$\mathcal{SYS} : \begin{cases} \dot{x}(t) = \begin{bmatrix} 0_{2\times 3} & \bar{A}(x^{(1)}(t)) \\ 0_{3\times 2} & 0_{3\times 3} \end{bmatrix} x(t) + \begin{pmatrix} \varphi^{(1)}(x^{(1)}(t)) \\ \varphi^{(2)}(x^{(1)}(t), x^{(2)}(t)) \end{pmatrix} \\ y^{(a)}(t) \ = Cx_{\bar{\tau}}(t) = x_{\bar{\tau}}^{(1)}(t) \end{cases} \tag{2.82}$$

with

$$\bar{A}(x^{(1)}) = \begin{bmatrix} -k_1 x_1^{(1)} & 1 & x_1^{(1)}\left(1 - \dfrac{x_1^{(1)} + x_2^{(1)}}{T_{\max}}\right) \\ k_1' x_1^{(1)} & 0 & 0 \end{bmatrix}, \varphi^{(1)}(x^{(1)}(t)) = \begin{bmatrix} -\mu_T x_1^{(1)}(t) \\ -\mu_I x_2^{(1)}(t) \end{bmatrix},$$
$$\varphi^{(2)}(x^{(1)}(t), x^{(2)}(t)) = \begin{bmatrix} N\mu_b x_2^{(1)}(t) - k_1 x_1^{(1)} x_1^{(2)}(t) - \mu_V x_1^{(2)}(t) \\ 0 \\ 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} I_2 & 0_{2,3} \end{bmatrix}. \tag{2.83}$$

As in the previous case, the output of the system is the variables $T(t)$ and $I(t)$ in the presence of multiple sampling and multiple time-varying delay. Having the following form:

$$y^{(a)}(t) = \begin{pmatrix} x_1^{(1)}(t - \bar{\tau}_1(t)) \\ x_2^{(1)}(t - \bar{\tau}_1(t)) \end{pmatrix} \tag{2.84}$$

with

$$\begin{cases} \bar{\tau}_1(t) & = t - t_1^{(k_1)} + \tau_1(t_1^{(k_1)}),\ \forall t \in \left[t_1^{(k_1)}, t_1^{(k_1+1)} \right[, k_1 \geq 0 \\ \bar{\tau}_2(t) & = t - t_2^{(k_2)} + \tau_2(t_2^{(k_2)}),\ \forall t \in \left[t_2^{(k_2)}, t_2^{(k_2+1)} \right[, k_2 \geq 0 \end{cases} \tag{2.85}$$

The objective is to obtain the continuous and actual estimation of the system (2.82), therefore an observer of the form (2.59) is designed. The initial conditions of the system and observer states are set as follows:

$$x(0) = \begin{bmatrix} 1000 & 0 & 10 \times 10^{-3} \end{bmatrix}^T, \hat{x}(s) = \begin{bmatrix} 1200 & 0.8 & 0.5 & 0 & 0 \end{bmatrix}^T \text{ with } s \in [-\tau_M, 0],$$

and the initial observer parameters are:

$$S(0) = I_{5\times5},\ \Delta_\theta = \begin{bmatrix} I_{2\times2} & 0_{2\times3} \\ 0_{3\times2} & \frac{1}{\theta} I_{3\times3} \end{bmatrix}, \theta = 1.2,$$

$$\text{and } g(s) = 0_{3\times1} \forall s \in [-\Delta_M - \tau_M, 0].$$

The first output is sampled with a constant sampling period $\Delta_1 = 0.01$ day and exhibits a time-varying delay $\tau_1 = 0.18(0.75 + 0.25\sin(0.2t))$ day. The second output presents a constant sampling period $\Delta_2 = 0.012$ day and the time-varying delay is $\tau_2(t) = 0.3(0.75 + 0.25\cos(0.4t))$ day. Taking into consideration (2.80), one obtains the upper limits as follows:

$$\begin{cases} \Delta_M = & \max\{\ \Delta_1,\ \ \Delta_2\ \} = \max\{\ 0.01,\ \ 0.12\ \} \text{ day } = 0.12 \text{ day}, \\ \tau_M = & \max\{\ \tau_1,\ \ \tau_2\ \} = \max\{\ 0.18,\ \ 0.3\ \} \text{ day } = 0.3 \text{ day}. \end{cases}$$

This upper bound of the involved delays is $\bar{\tau}_M = \Delta_M + \tau_M = 0.42$ day. The state variables estimated by the observer are shown in Figure 2.11-2.15. The estimation of the concentration of healthy CD4$^+$ T-cells is shown in Figure 2.11. The estimation of infected CD4$^+$ T-cells is shown in Figure 2.12. It is important to note that these two state variables are the outputs of the system and are sampled and delayed, so it is important to note that the observer achieves continuous and actual estimation of these variables. The estimation of free HIV cells in the human body is shown in Figure 2.13. The estimation of the s parameter shown in Figure 2.14 helps us to know how the uninfected CD4$^+$ T-cells are being produced. The increase of this value indicates that the infected person is producing more healthy cells, so the infected cells decrease. The estimation of the r parameter is shown in Figure 2.15. This parameter tells us how fast the population of healthy cells can grow when infected cells are present and is also an indicator of how infected cells affect the ability of the immunologic system to control the infection. Therefore, the monitoring of these parameters is essential to improve the prediction of disease progression and the design of therapeutic strategies. In order to show that the estimation error converges to zero even in the presence of multiple samples and multiple delays. Figure 2.16, the absolute values of the error in each variable and the parameters are shown.

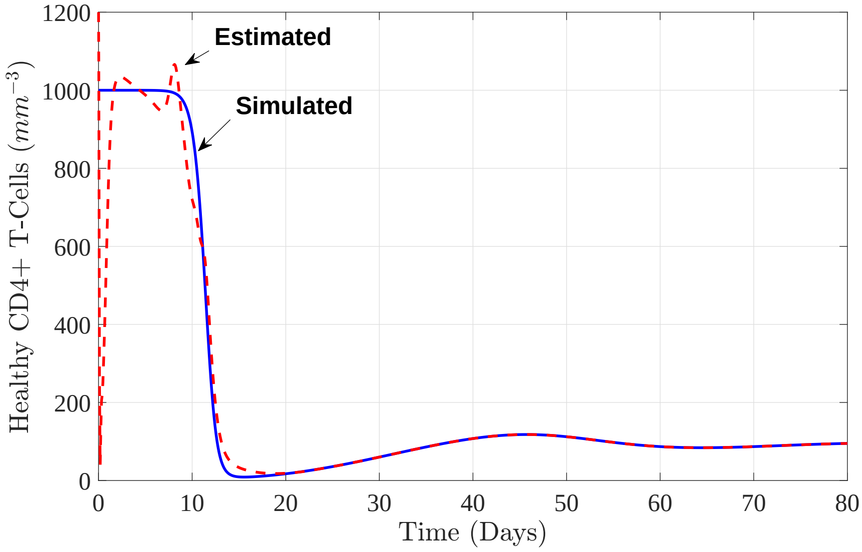

Figure 2.11: Estimation of the concentration of healthy CD4$^+$ T-cells by the observer with unknown parameters

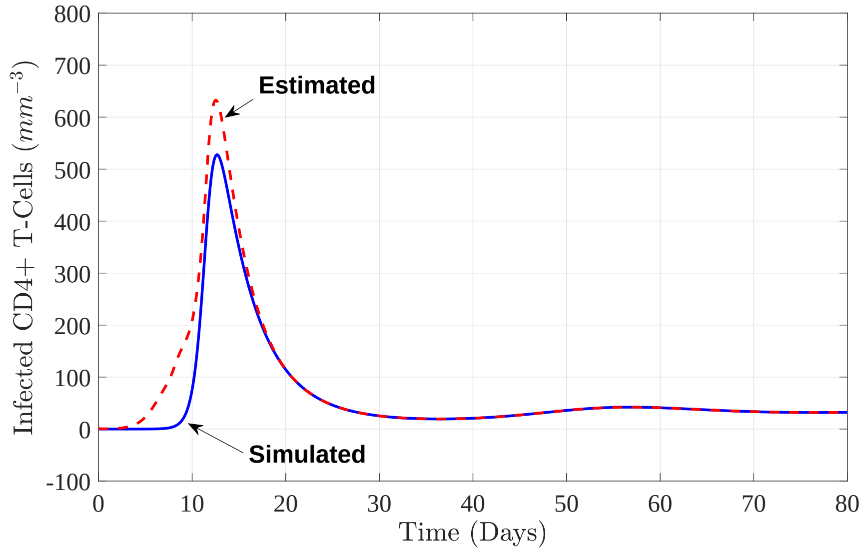

Figure 2.12: Estimation of the concentration of infected CD4$^+$ T-cells by the observer with unknown parameters

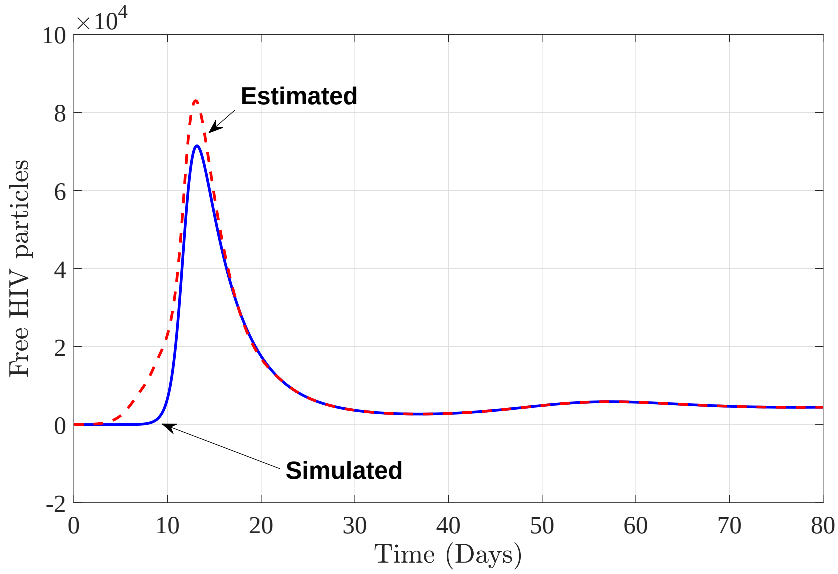

Figure 2.13: Estimation of the free HIV particles by the observer with unknown parameters

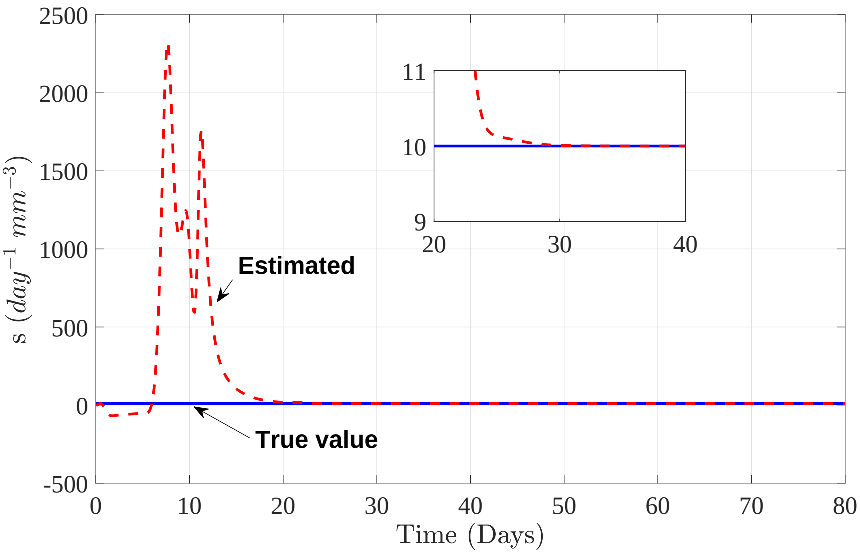

Figure 2.14: Estimation of the unknown parameter s by the observer

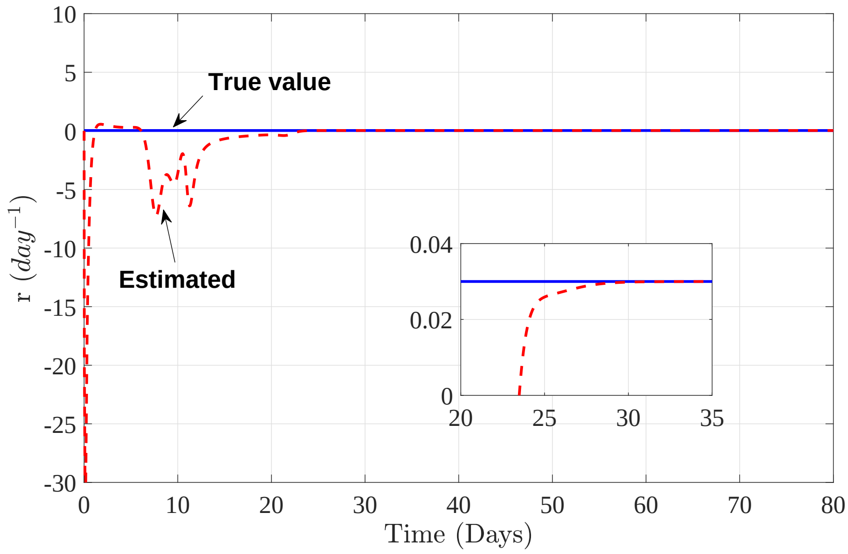

Figure 2.15: Estimation of the unknown parameter r by the observer

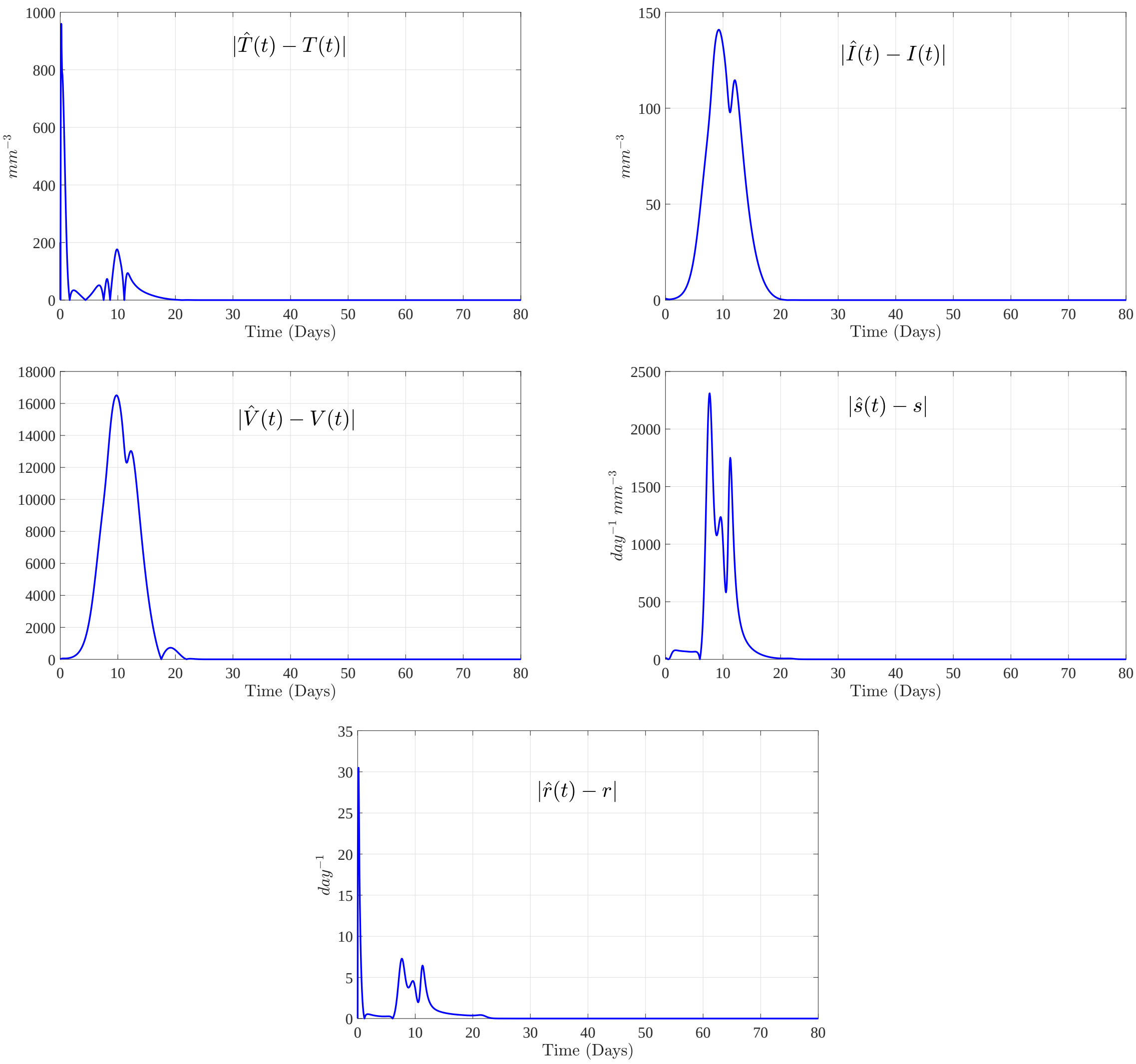

Figure 2.16: Absolute error between each estimated state and its original state

2.6 Conclusion

This chapter presented the design of a high gain observer for a non-uniformly observable system considering that the outputs of the system are only present at time instants and have time delays. This observer is first designed considering that the outputs have the same sampling instant and a time-varying delay to present the structure of the observer and the technical notions. Subsequently, the observer is redesigned to deal with the practical case, which is the multiple presence of delays and multiple samples in each of its outputs. An important emphasis is made on the observability conditions to be met by the system and the conditions on the maximum sampling and delay values. It is properly demonstrated that the observer error will converge to zero and that the decay rate is related to the upper bounded values of sampling and time-delay. The performance and applicability of the proposed observer have been demonstrated by means of a biological system. It exploits the class of non-uniformly observable systems to jointly estimate unknown states and parameters in the system.

Chapter 3

Observer Synthesis for a class of nonlinear system under unknown time-delay in the output

3.1 Introduction

This chapter deals with the design of an observer for a class of nonlinear systems in the presence of unknown delay in the output signal. This presents an interesting challenge, since in practice an approximate value of the delay can be known under optimal conditions; however, this can change and what was previously designed degrades. Within the observer design, if the delay is not known exactly, the estimation of the current states will be bounded to a bounded region. Therefore, to eliminate this region, it is necessary to know its value with accuracy.

The observer to be designed is considered with a cascade structure. The main contribution is in the observer head, since it will be in charge of estimating the value of the unknown delay. Subsequently, the predictors should be able to estimate the current vector of the system.

The performance of the proposed cascaded observer is highlighted by simulations on an academic example and a robotic arm.

3.2 Problem statement

Let us consider the following class of nonlinear systems with unknown time-delay in the output:

$$\mathcal{SYS}: \begin{cases} \dot{x}(t) = Ax(t) + \varphi(u(t), x(t)) \\ y_\tau(t) = Cx(t-\tau) = x_1(t-\tau) \end{cases} \tag{3.1}$$

where:

$$A = \begin{bmatrix} 0_{(n-1),1} & I_{(n-1)} \\ 0 & 0_{1,(n-1)} \end{bmatrix}, \varphi(x(t), u(t)) = \begin{pmatrix} \varphi_1(u(t), x_1(t)) \\ \varphi_2(u(t), x_1(t), x_2(t)) \\ \vdots \\ \varphi_{(n-1)}(u(t), x_1(t), \ldots, x_{(n-1)}(t)) \\ \varphi_n(u(t), x(t)) \end{pmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{1\times n} \tag{3.2}$$

where $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$ denotes the state of the system; $y_\tau(t) \in \mathbb{R}^p$ is the delayed output; $\tau > 0$ is the time-delay and it is assumed to be unknown; $\varphi(u(t), x(t)$ is a nonlinear vector function and each k^{th}, $k = 1, \ldots, n$ is assumed to have a triangular structure with respect to the state x, i.e.

$$\varphi_k(u, x) = \varphi_k(u, \bar{x}_k) \text{ where } \bar{x}_k = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \in \mathbb{R}^k \tag{3.3}$$

The following assumptions are necessary for the observer design.

Assumption 3.2.1 *The state $x(t)$ and the system input $u(t)$ are bounded, i.e., $x(t) \in X$ and $u \in U$ where $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}$ are compact sets. More precisely, there exists a positive constant ρ_M, such that $\forall t \geq 0, |x_i(t)| \leq \rho_M, i = 1, \ldots, n$.*

Assumption 3.2.2 *Each function $\varphi^{(i)}, i = 1, \ldots, n$ is Lipschitz on X with respect to x and uniform in u, i.e.,*

$$\exists L_\varphi > 0;\ \forall u \in U; \forall (x, \bar{x}) \in X \times X:$$

$$\|\varphi_i(u, x) - \varphi_i(u, \bar{x})\| \leq L_\varphi \sum_{k=1}^{i} |x_k - \bar{x}_k|. \tag{3.4}$$

Assumption 3.2.3 *The input $u(t)$ is differentiable with respect to time.*

Assumption 3.2.4 *The unknown time-delay $\tau > 0$ is a positive constant and is assumed to be arbitrarily large.*

Before proceeding with the observer design, let us consider the following saturation function:

$$\forall \xi \in \mathbb{R}, sat_r(\xi) \quad = \quad r \cdot sat(\xi/r) = \begin{cases} \xi & \text{if} \quad |\xi| \leq r, \\ r \cdot sign(\xi) & \text{if} \quad |\xi| \geq r, \end{cases} \tag{3.5}$$

where r is a positive real and $sign(\cdot)$ is the usual signum function. This allows to saturate the nonlinearities $\varphi_k' s$ for $k = 1, \ldots, n$ as follows

$$\varphi_i^s(u, x) = \varphi_i(u, sat_R(x_1), \ldots, sat_R(x_i)) \tag{3.6}$$

with

$$R > \rho_M, \tag{3.7}$$

where ρ_M is the positive real defined in assumption **A1**. Bearing in mind that the Lipschitz constant of the saturation function is equal to one, one gets

$$\begin{aligned} &\forall u \in U;\ \forall (\hat{x}, x) \in \mathbb{R}^n \times \mathbb{R}^n, \\ &\|\varphi^s u(t), x(t) - \varphi^s(u, \bar{x})\| \leq L_\varphi \|x - \bar{x}\| \end{aligned} \tag{3.8}$$

Now, the system (3.1) can be rewritten as follows:

$$\mathcal{SYS} : \begin{cases} \dot{x}(t) = Ax(t) + \varphi^s(u(t), x(t)) \\ y_\tau(t) = x_1(t - \tau) \end{cases} \tag{3.9}$$

It is important to note that the observer will be designed for the system (3.9). However, the observer will estimate the trajectories of the system (3.1) since both lie in X.

The main goal is to achieve an estimation of the actual state vector of the system (3.1). For this purpose, first, the value of the unknown delay present in the output signal must be estimated, and then, with this estimation, this delay must be removed from the system. Inspired by the structure of the cascade observer, which consists of an observer in the head and m subsystems called predictors, our aim is to achieve a joint estimate of the value of the unknown delay and the actual state vector of the system (3.1).

3.3 Head observer design

In essence the head of the cascade estimates the delayed state vector of the system; however, in this case it must also estimate the value of the unknown time delay present in the system. In order to achieve this, one proceeds to define the new state vector $z(t)$ as the delayed version of the system (3.9), i.e $z(t) = x(t-\tau)$. Thus, the delayed system is defined as follows:

$$\mathcal{SYS}_\tau : \begin{cases} \dot{z}(t) = Az(t) + \varphi^s(u(t-\tau), z(t)) \\ \bar{y}(t) = Cz(t) = z_1(t) \end{cases} \tag{3.10}$$

where $z(t) = [z_1, z_2, \dots, z_n]^T \in \mathbb{R}^n$ is the actual state vector of the system (3.10) and it is the delayed version of the system (3.9). The matrix A and C are the same as denoted by (3.2). Note that the actual output signal $\bar{y}(t)$ of the system (3.10) corresponds to the available delayed output of the system (3.9), i.e. $\bar{y}(t) = y_\tau(t)$. Focusing on the nonlinear vector function $\varphi^s(u(t-\tau), z(t))$ one can see the presence of the unknown term (τ), this results in a nonlinear parameterization. Therefore, to achieve the estimation of the unknown parameter and the state vector of the system (3.10), an adaptive observer inspired by Farza et al. (2018c).

Before presenting the equations of the adaptive observer it is important to note that the Assumptions 3.2.1-3.2.3 are fulfilled satisfied and the following assumption is introduced:

Assumption 3.3.1 *The nonlinearity $\varphi^s(u(t-\tau), z(t))$ is differentiable with respect to u and satisfies $\frac{\partial}{\partial u}\varphi^s(u(t-\tau), z) \neq 0$. Moreover, for each $i = 1, \dots, n$ such that $\frac{\partial}{\partial u}\varphi^s_i(u(t-\tau), z) \neq 0$.*

According to Assumption 3.2.4, let us define the term $\nu \in 1, \dots, n$ as follows:

$$\nu = \min\left\{k \in 1, \dots, n \ s.t \ \frac{\partial}{\partial u}\varphi_k(u, \underline{z}_k) \neq 0\right\} \tag{3.11}$$

Bearing in mind the above, it is possible to establish the following observer for the state estimation of the system (3.10). The equations are the following:

$$\mathcal{OBS} : \begin{cases} \dot{\hat{z}}(t) & = A\hat{z}(t) + \varphi^s(u(t-\hat{\tau}), \hat{z}(t)) \\ & \quad -\theta\Delta_\theta^{-1}\left(S^{-1} + p(t)\Upsilon(t)\Upsilon^T(t)\right)C^T(C\hat{z}(t) - y_\tau(t)), \\ \dot{\hat{\tau}}(t) & = -\theta^{\nu+1}p(t)\Upsilon^T(t)C^T(C\hat{z}(t) - y_\tau(t)) \\ \dot{\Upsilon}(t) & = \theta\left(A - S^{-1}C^TC\right)\Upsilon + \theta B_\nu\Phi(u(t-\hat{\tau}), z), \Upsilon(0) = 0 \\ \dot{p}(t) & = -\theta p^2(t)\Upsilon^T(t)C^TC\Upsilon(t) + \theta p(t), \ P(0) > 0 \end{cases} \tag{3.12}$$

where $\hat{z}(t) \in \mathbb{R}^n$ is the estimate state vector and $\hat{\tau} \in \mathbb{R}$ the estimated unknown time delay, y_τ is the delayed output of the system (3.1). The matrix A and C are given by (3.2). The diagonal matrix Δ_θ is defined as follows:

$$\Delta_\theta = diag\left[1,\ \frac{1}{\theta},\ \ldots,\ \frac{1}{\theta^{n-1}}\right], \tag{3.13}$$

where $\theta > 0$ is a positive real design parameter. The matrix S is the unique solution of the following Lyapunov equation:

$$S + A^T S + SA - C^T C = 0. \tag{3.14}$$

Recall that the matrix S is symmetric and positive definite and it is such that the eigenvalues of $A - S^{-1}C^T C$ are all located at -1, as Farza et al. (2018c).

The function $\Phi(u(t-\hat{\tau}), \hat{z})$ is defined as follows:

$$\Phi(u(t-\hat{\tau}), \hat{z}) = -\frac{\partial}{\partial u}\varphi^s(u(t-\hat{\tau}(t)), \underline{\hat{z}_\nu})\dot{u}(t-\hat{\tau}(t)), \tag{3.15}$$

and B_ν is a column vector with all entries equal to zero excepting the ν'th one which is equal to 1, i.e.

$$B_\nu(\nu) = 1 \text{ and } \forall i = 1, \ldots, n \setminus \{\nu\}, B_\nu(i) = 0. \tag{3.16}$$

In order to ensure observer convergence, the following assumptions need to be defined.

Assumption 3.3.2 *The input u is such that for any trajectory $\hat{z}(t)$ of system (3.12) starting from $\hat{z}(0) \in X$, the matrix $C\Upsilon(t)$ is persistently exciting in the following sense*

$$\begin{gathered} \forall T > 0, \exists \delta > 0; \forall \theta \geq \theta_0; \forall t > \frac{1}{\theta}, \\ \int_{t-\frac{1}{\theta}}^{t} \Upsilon^T(s) C^T C \Upsilon(s) ds \leq \delta \end{gathered} \tag{3.17}$$

Under the Assumption 3.3.2, one can show that the real valued function $p(t)$ given by (3.12) is positive and bounded from below when $p(0)$, i.e.

$$\begin{gathered} \exists \underline{p}, \bar{p} > 0; \forall t \geq 0, \\ \underline{p} \leq p(t) \leq \bar{p}. \end{gathered} \tag{3.18}$$

The nonlinear function $\varphi^s(u(t-\hat{\tau}), \hat{z})$ satisfies the following inequality

$$\begin{gathered} \exists \kappa \in (0,1); \forall z \in X, \forall \bar{\tau}, \tau \in \mathbb{R}; \forall t \geq 0, \\ \left| \frac{\partial}{\partial \tau}\varphi^s(u(t-\bar{\tau}), \underline{z_\nu}) - \frac{\partial}{\partial \tau}\varphi^s(u(t-\tau), \underline{z_\nu}) \right| \leq \kappa \sqrt{\frac{\underline{p}}{\lambda_M}}, \end{gathered} \tag{3.19}$$

where $\underline{p}$ is the lower bound of p and λ_M is the largest eigenvalue of the SPD matrix S defined as in (3.14).

3.4 Cascade predictors design

The adaptive observer (3.12) can be used to ensure the convergence of the auxiliary system (3.9) and the value of the time-delay τ. The objective is to propose a structure to estimate the system (3.1). For this, a cascade structure is proposed Farza et al. (2018a); Tréangle et al. (2018). Thus, one defines the following states and inputs of the predictors:

$$x_j(t) = x\left(t - \hat{\tau} + \frac{j}{m}\hat{\tau}\right),\ u_j(t) = u\left(t - \hat{\tau} + \frac{j}{m}\hat{\tau}\right),\ \text{ for } j = 0, \ldots, m, \forall t \geq -j\frac{\tau}{m} \quad (3.20)$$

where m is the number of predictors in the cascade.

Notice that in the position $j = 0$, the state to predict is $x_0(t)$, this state $x_0(t)$ denotes the delayed state with τ, i.e., $z(t) = x(t - \tau) \triangleq x_0(t)$, and it has been estimated by (3.12).

The objective of each predictor in the cascade is to predict the state of the preceding subsystem over a prediction horizon equal to $\frac{\hat{\tau}}{m}$, where the last predictor estimates the actual state vector. The following properties are satisfied through the cascade predictors:

$$x_j\left(t - \frac{\hat{\tau}}{m}\right) = x_{j-1}(t) \text{ and } u_j\left(t - \frac{\hat{\tau}}{m}\right) = u_{j-1}(t). \quad (3.21)$$

Bearing in mind the above and miming the structure of system (3.9), one can propose a dynamical system of each $x_j(t), j = 1, \ldots, m$ as follows:

$$\dot{x}_j(t) = Ax_j(t) + \varphi^{(s)}(u_j(t), x_j(t)). \quad (3.22)$$

The above system can be rewritten as follows:

$$\dot{x}_j(t) = \bar{A}x_j(t) + \varphi^{(s)}(u_j(t), x_j(t)) + (A - \bar{A})x_j(t) \quad (3.23)$$

where $\bar{A}$ is Hurwitz and it is a parameter in the cascade.

The state $x_j(t)$ given by (3.23) can be expressed as follows:

$$\begin{aligned} x_j(t) &= e^{\bar{A}\frac{\hat{\tau}}{m}} x_j\left(t - \frac{\hat{\tau}}{m}\right) + \int_{t-\frac{\hat{\tau}}{m}}^{t} e^{\bar{A}(t-s)}\left(\varphi^{(s)}(u_j(s), x_j(s)) + (A - \bar{A})x_j(s)\right) ds \\ &= e^{\bar{A}\frac{\hat{\tau}}{m}} x_{j-1}(t) + \int_{t-\frac{\hat{\tau}}{m}}^{t} e^{\bar{A}(t-s)}\left(\varphi^{(s)}(u_j(s), x_j(s)) + (A - \bar{A})x_j(s)\right) ds \end{aligned} \quad (3.24)$$

Now, one proceeds to design an observer to predict state $x_j(t)$ given by (3.22). Miming the structure of the system, the predictor is given by:

$$\dot{\hat{x}}_j(t) = A\hat{x}_j(t) + \varphi^{(s)}(u_j(t), \hat{x}_j(t)) - G_j(t) \quad (3.25)$$

where $G_j(t),\ j=1,\dots,m$ is the correction term in each predictor. Each function $G_j(t)$ must be chosen in order to guarantee the exponential convergence to zero of the prediction error.

Now, the equation (3.25) can be rewritten as follows:

$$\dot{\hat{x}}_j(t) = \bar{A}\hat{x}_j(t) + \varphi^{(s)}(u_j(t), \hat{x}_j(t)) + \left(A - \bar{A}\right)\hat{x}_j(t) - G_j(t) \tag{3.26}$$

As the predictor system given by (3.24), the prediction $\hat{x}_j(t)$ can be expressed as follows:

$$\hat{x}_j(t) = e^{\bar{A}\frac{\hat{\tau}}{m}}\hat{x}_j\left(t - \frac{\hat{\tau}}{m}\right) + e^{\bar{A}t}\int_{t-\frac{\hat{\tau}}{m}}^{t} e^{\bar{A}s}\left(\varphi^{(s)}(u_j(s), x_j(s)) + (A-\bar{A})x_j(s) - G_j(s)\right)ds \tag{3.27}$$

Considering two successive predictors $\hat{x}_j$ and $\hat{x}_{j-1}$, one obtains the following:

$$\hat{x}_j(t) = e^{\bar{A}\frac{\hat{\tau}}{m}}\hat{x}_{j-1}(t) + r_j(t) + e^{\bar{A}t}\int_{t-\frac{\hat{\tau}}{m}}^{t} e^{\bar{A}s}\left(\varphi^{(s)}(u_j(s), x_j(s)) + (A-\bar{A})x_j(s)\right)ds \tag{3.28}$$

where the functions $r_j^{'}s,\ j=1,\dots,m$ are differentiable with respect to time and which shall be determined simultaneously with the correction terms $G_j^{'}s$ such that the relation (3.28) holds true for any $t \geq 0$.

Determination of the predictor gains:

The determination of predictor gains G_j is presented below. Subtracting (3.28) from equation (3.27), one obtains:

$$e^{\bar{A}\frac{\hat{\tau}}{m}}\left(\hat{x}_j\left(t-\frac{\hat{\tau}}{m}\right) - \hat{x}_{j-1}(t)\right) - r_j(t) = e^{\bar{A}t}\int_{t-\frac{\hat{\tau}}{m}}^{t} e^{-\bar{A}s}G_j(s)ds$$

Differentiating with respect to time each side of the above equation, one gets:

$$\begin{aligned}
e^{(\bar{A}\frac{\hat{\tau}}{m})}\left(\dot{\hat{x}}_j\left(t-\frac{\hat{\tau}}{m}\right) - \dot{\hat{x}}_{j-1}(t)\right) - \dot{r}_j(t) &= G_j(t) - e^{(\bar{A}\frac{\hat{\tau}}{m})}G_j\left(t-\frac{\hat{\tau}}{m}\right) \\
&\quad + \bar{A}\int_{t-\frac{\hat{\tau}}{m}}^{t} e^{\bar{A}(t-s)}G_j(s)ds \\
&= \bar{A}e^{\bar{A}\frac{\hat{\tau}}{m}}\left(\hat{x}_j\left(t-\frac{\hat{\tau}}{m}\right) - \hat{x}_{j-1}(t)\right) - \bar{A}r_j(t) \\
&\quad + G_j(t) - e^{\bar{A}\frac{\hat{\tau}}{m}}G_j\left(t-\frac{\hat{\tau}}{m}\right)
\end{aligned} \tag{3.29}$$

Substituting in the above equation the expression of $\dot{\hat{x}}_{j-1}(t)$ and $\dot{\hat{x}}_j\left(t-\frac{\hat{\tau}}{m}\right)$ given by (3.24) yields:

$$e^{(\bar{A}\frac{\hat{\tau}}{m})}\left(\dot{\hat{x}}_j\left(t-\frac{\hat{\tau}}{m}\right)-\dot{\hat{x}}_{j-1}(t)\right)+\varphi^{(s)}\left(u_{j-1}(t),\hat{x}_j\left(t-\frac{\hat{\tau}}{m}\right)\right)-\varphi^{(s)}\left(u_{j-1}(t),\hat{x}_{j-1}(t)\right)$$
$$+G_{j-1}(t)-G_j\left(t-\frac{\hat{\tau}}{m}\right)=\bar{A}e^{\bar{A}\frac{\hat{\tau}}{m}}\left(\hat{x}_j\left(t-\frac{\hat{\tau}}{m}\right)-\hat{x}_{j-1}(t)\right)+G_j(t)$$
$$-e^{\bar{A}\frac{\hat{\tau}}{m}}G_j\left(t-\frac{\hat{\tau}}{m}\right)+\left(\dot{r}_j(t)-\bar{A}r_j(t)\right) \quad (3.30)$$

Considering that the matrices $\bar{A}$ and $e^{\bar{A}\frac{\hat{\tau}}{m}}$ commute with each other, the equation above leads to

$$G_j(t)=e^{\bar{A}\frac{\hat{\tau}}{m}}\left(G_{j-1}(t)+(A-\bar{A})\left(\hat{x}_j\left(t-\frac{\hat{\tau}}{m}\right)-\hat{x}_{j-1}(t)\right)\right.$$
$$\left.+\varphi^{(s)}\left(u_{j-1}(t),\hat{x}_j\left(t-\frac{\hat{\tau}}{m}\right)\right)-\varphi^{(s)}\left(u_{j-1}(t),\hat{x}_{j-1}(t)\right)\right)-\left(\dot{r}_j(t)-\bar{A}r_j(t)\right) \quad (3.31)$$

Now, if one chooses $r_j(t)$ such that

$$\dot{r}_j(t)=\bar{A}r_j(t) \quad (3.32)$$

The correction term G_j results in

$$G_j(t)=e^{\bar{A}\frac{\hat{\tau}}{m}}\left(G_{j-1}(t)+(A-\bar{A})\left(\hat{x}_j\left(t-\frac{\hat{\tau}}{m}\right)-\hat{x}_{j-1}(t)\right)\right.$$
$$\left.+\varphi^{(s)}\left(u_{j-1}(t),\hat{x}_j\left(t-\frac{\hat{\tau}}{m}\right)\right)-\varphi^{(s)}\left(u_{j-1}(t),\hat{x}_{j-1}(t)\right)\right) \quad (3.33)$$

Recall that since the matrix $\bar{A}$ is Hurwitz, there exists a positive number $\beta\geq 1$ such that (Farza et al., 2018b; Hernández-González et al., 2016)

$$\forall t\geq 0:\|e^{\bar{A}t}\|\leq\beta e^{-\bar{a}t}, \quad (3.34)$$

with $\bar{a}=\min_{i\in\{1,\ldots,n\}}|\Re(\lambda_i(\bar{A}))|$ where $\lambda_i(\bar{A}), i=1,\ldots,n$ are the n eigenvalues of $\bar{A}$ (with negative real parts).

The equations of the cascade predictors are as follows:

$$\begin{cases}\dot{\hat{x}}_j(t)=A\hat{x}_j(t)+\varphi^{(s)}(u_j(t),\hat{x}_j(t))-G_j(t)\\ G_j(t)=e^{\bar{A}\frac{\hat{\tau}}{m}}\left(G_{j-1}(t)+(A-\bar{A})\left(\hat{x}_j(t-\frac{\hat{\tau}}{m})-\hat{x}_{j-1}(t)\right)\right.\\ \qquad\left.+\varphi^{(s)}(u_{j-1}(t),\hat{x}_j(t-\frac{\hat{\tau}}{m}))-\varphi^{(s)}(u_{j-1}(t),\hat{x}_{j-1}(t))\right)\end{cases} \quad (3.35)$$

where $\hat{x}_j(t)\in\mathbb{R}^n$ is the estimated state of $x_j(t)$ for $j=1,\ldots,m$, $\bar{A}\in\mathbb{R}^{n\times n}$ is a Hurwitz matrix such as $\bar{A}=A-\lambda I_n$ with $\lambda>0$ is a design parameter in the cascade. Notice that

$\hat{\tau}$ is the value of unknown time-delay estimated by the head observer given by (3.12).

In summary, the estimation of the actual state vector under unknown time-delay can be handled by the following cascade dynamical systems:

$$\left\{\begin{array}{rl} \text{For } j = & 0 \\ \dot{\hat{x}}_0(t) & = A\hat{x}_0(t) + \varphi^s(u(t-\hat{\tau}), \hat{x}_0(t)) - G_0(t), \\ G_0(t) & = \theta\Delta_\theta^{-1}\left(S^{-1} + p(t)\Upsilon(t)\Upsilon^T(t)\right)C^T(C\hat{x}_0(t) - y_\tau(t)), \\ \dot{\hat{\tau}}(t) & = -\theta^{\nu+1}p(t)\Upsilon^T(t)C^T(C\hat{x}_0(t) - y_\tau(t)), \\ \dot{\Upsilon}(t) & = \theta\left(A - S^{-1}C^TC\right)\Upsilon(t) + \theta B_\nu\Phi(u(t-\hat{\tau}), \hat{x}_0(t)), \\ \dot{p}(t) & = -\theta p^2(t)\Upsilon^T(t)C^TC\Upsilon(t) + \theta p(t) \\ \text{For } j = & 1, \dots, m \\ \dot{\hat{x}}_j(t) & = A\hat{x}_j(t) + \varphi^{(s)}(u_j(t), \hat{x}_j(t)) - G_j(t), \\ G_j(t) & = e^{\bar{A}\frac{\hat{\tau}}{m}}\left(G_{j-1}(t) + (A-\bar{A})\left(\hat{x}_j(t-\frac{\hat{\tau}}{m}) - \hat{x}_{j-1}(t)\right)\right. \\ & \left. \quad +\varphi^{(s)}(u_{j-1}(t), \hat{x}_j(t-\frac{\hat{\tau}}{m})) - \varphi^{(s)}(u_{j-1}(t), \hat{x}_{j-1}(t))\right) \end{array}\right. \tag{3.36}$$

where $\hat{x}_j(t) \in \mathbb{R}^n$ is the estimated state of $x_j(t)$ for $j = 0, \dots, m$, Δ_θ is given by (3.13), $\theta > 0$ is adjustment design, $y_\tau(t)$ is the available output of the system (3.1), $\bar{A} = A - \lambda I_n \in \mathbb{R}^{n\times n}$ and $\lambda > 0$ are design parameters in the cascade.

The above observer delay equations (3.20) are initialized as follows:

$$\hat{x}(0) = \hat{x}(-\hat{\tau}(0)) \text{ and } \hat{x}_j\left(s - \hat{\tau}(0) + j\frac{\hat{\tau}(0)}{m}\right), s \in \left[-\frac{\hat{\tau}(0)}{m}\right], j = 1, \dots, m, \tag{3.37}$$

where $\hat{x}(s), s \in [-\tau(0), 0]$ is any a *priori* selected state vector and $\tau(0) > 0$ must be a positive constant.

Now, the following theorem can be proposed.

Theorem 3.4.1 *Consider system (3.1) subject to Assumption* ***A1-A6*** *together with the cascade observer (3.12). If the matrix $\bar{A}$ is chosen such that $\bar{a} \leq \frac{\theta}{2}$ and if the number of predictors m is selected such that*

$$\varrho\frac{\hat{\tau}}{m} \leq 1 \tag{3.38}$$

where

$$\varrho = \beta\left(L_\varphi + \|A - \bar{A}\|\right) \tag{3.39}$$

with β is given in (3.34) and L_φ is the Lipschitz constant of φ defined in (3.8). Then, one has for $j = 1, \ldots, m$:

$$\|\tilde{x}_j(t)\| \leq \rho_j e^{-\bar{a}t},\ t \geq 0 \tag{3.40}$$

where

$$\rho_j = \frac{\varrho}{1 - \varrho\frac{\hat{\tau}}{m}} \int_{-\frac{\hat{\tau}}{m}}^{0} \|\tilde{x}_j(s)\| ds + \beta\zeta_m^j \sigma \left(\|\tilde{x}_0(0)\| + (1-\delta)\|\tilde{\tau}(0)\|\right) + \frac{\beta}{1 - \varrho\frac{\hat{\tau}}{m}} \sum_{k=0}^{j-1} \zeta_m^k \|r_{j-k}(0)\| \tag{3.41}$$

where the functions $r_k^{'}s$, $k = 1, \ldots, m$ are given by (3.32), σ is the conditioning number of matrix S and

$$\zeta_m = \frac{e^{-\bar{a}\frac{\hat{\tau}}{m}}}{1 - \varrho\frac{\hat{\tau}}{m}}. \tag{3.42}$$

In the next subsection, the proof of the theorem will be presented. It is important to note that the error of each predictor and of the observer at the head of the cascade will converge exponentially to zero. Moreover, the estimation of the unknown delay present in the output of the original system is achieved. Therefore, the original state of the delay-free system is estimated with the last predictor of the cascade, i.e. in the predictor $j = m$.

3.4.1 Proof of Theorem 3.4.1

First, one proceeds to analyze the head of the observer in the cascade, i.e. $j = 0$. Let us define $\tilde{x}_0(t) = \hat{x}_0(t) - x_0(t)$ as the observation error in states.
According to (3.10) and (3.12), one has:

$$\begin{aligned}
\dot{\tilde{x}}_0(t) =& A\hat{x}_0(t) + \varphi^s(u(t-\hat{\tau}), \hat{x}_0(t)) - \theta\Delta_\theta^{-1} p(t)\Upsilon(t)\Upsilon^T(t)C^T C\tilde{x}_0(t) \\
& - \left[Ax_0(t) + \varphi^s(u(t-\tau), x_0(t))\right], \\
\dot{\tilde{x}}_0(t) =& (A - \theta\Delta_\theta^{-1} S^{-1} C^T C)\tilde{x}_0(t) + \varphi^s(u_0(t), \hat{x}_0(t), \hat{\tau}(t), x_0(t), \tau(t)) \\
& - \theta\Delta_\theta^{-1} p(t)\Upsilon(t)\Upsilon^T(t)C^T C\tilde{x}_0(t)
\end{aligned} \tag{3.43}$$

and defining $\tilde{\tau}(t) = \hat{\tau}(t) - \tau$, one gets:

$$\begin{aligned}
\dot{\tilde{\tau}}(t) =& \dot{\hat{\tau}}(t) - \dot{\tau}, \\
\dot{\tilde{\tau}}(t) =& -\theta^{\nu+1} p(t)\Upsilon^T(t)C^T C\tilde{x}_0(t)
\end{aligned} \tag{3.44}$$

According to Assumption 3.2.4 the time derivative of τ is equal to zero, i.e $\dot{\tau} = 0$. Let us define $\tilde{\varphi}^s(u_0(t), \hat{x}_0(t), \hat{\tau}(t), x_0(t), \tau(t))$ as follows:

$$\begin{aligned}\tilde{\varphi}^s(u_0(t), \hat{x}_0(t), \hat{\tau}(t), x_0(t), \tau(t)) &= \varphi^s(u(t-\hat{\tau}), \hat{x}_0(t)) - \varphi^s(u(t-\tau), x_0(t)) \\ &\triangleq \tilde{\varphi}^{s_1}(u_0(t), \hat{\tau}(t), \tau(t), \hat{x}_0(t)) + \tilde{\varphi}^{s_2}(u_0(t), \tau(t), \hat{x}_0(t), x_0(t))\end{aligned} \tag{3.45}$$

with:

$$\begin{aligned}\tilde{\varphi}^{s_1}(u_0(t), \hat{\tau}(t), \tau(t), \hat{x}_0(t)) &= \varphi^s(u(t-\hat{\tau}), \hat{x}_0(t)) - \varphi^s(u(t-\tau), \hat{x}_0(t)) \\ \tilde{\varphi}^{s_2}(u_0(t), \tau(t), \hat{x}_0(t), x_0(t)) &= \varphi^s(u(t-\tau), \hat{x}_0(t)) - \varphi^s(u(t-\tau), x_0(t))\end{aligned} \tag{3.46}$$

Now, set the exponential error $\bar{x}_0(t) = \Delta_\theta \tilde{x}_0(t)$ and $\bar{\tau}(t) = \frac{1}{\theta^\nu}\tilde{\tau}(t)$, one gets:

$$\begin{aligned}\dot{\bar{x}}_0(t) &= \theta(A - \Delta_\theta^{-1} S^{-1} C^T C)\bar{x}_0(t) + \Delta_\theta \tilde{\varphi}^{s_1}(u_0(t), \hat{\tau}(t), \tau(t), \hat{x}_0(t)) \\ &\quad + \Delta_\theta \tilde{\varphi}^{s_2}(u_0(t), \tau(t), \hat{x}_0(t), x_0(t)) - \theta p(t)\Upsilon(t)\Upsilon^T(t) C^T C \bar{x}_0(t) \end{aligned} \tag{3.47}$$

$$\dot{\bar{\tau}}(t) = -\theta p(t)\Upsilon^T(t) C^T C \bar{x}_0(t). \tag{3.48}$$

The equation (3.47) can be expressed as follows:

$$\begin{aligned}\dot{\bar{x}}_0(t) &= \theta(A - \Delta_\theta^{-1} S^{-1} C^T C)\bar{x}_0(t) + \Delta_\theta \tilde{\varphi}^{s_1}(u_0(t), \hat{\tau}(t), \tau(t), \hat{x}_0(t)) \\ &\quad + \Delta_\theta \tilde{\varphi}^{s_2}(u_0(t), \tau(t), \hat{x}_0(t), x_0(t)) + \Upsilon(t)\dot{\bar{\tau}}(t)\end{aligned} \tag{3.49}$$

Let $\tilde{\varphi}^{s_1}_\nu$ be the ν'th component of $\tilde{\varphi}^{s_1}$ where ν is defined as in (3.11). Using the mean value theorem, one gets

$$\tilde{\varphi}^{s_1}(u_0(t), \hat{\tau}(t), \tau(t), \hat{x}_0(t)) = \frac{\partial}{\partial \tau}\varphi^s_\nu(u(t-\bar{\tau}), \underline{\hat{x}}_{0,\nu}(t))\tilde{\tau}(t) \tag{3.50}$$

for some unknown $\bar{\tau} \in \mathbb{R}$.

Adding and subtracting $\frac{\partial}{\partial \tau}\varphi^s_\nu(u(t-\bar{\tau}), \underline{\hat{x}}_{0,\nu})\tilde{\tau}(t)$ to the right side of the last equality, one gets

$$\begin{aligned}\tilde{\varphi}^{s_1}(u_0(t), \hat{\tau}(t), \tau(t), \hat{x}_0(t)) &= \frac{\partial}{\partial \tau}\varphi^s_\nu(u(t-\bar{\tau}), \underline{\hat{x}}_{0,\nu}(t))\tilde{\tau}(t) \\ &\quad + \left(\frac{\partial}{\partial \tau}\varphi^s_\nu(u(t-\bar{\tau}), \underline{\hat{x}}_{0,\nu}(t)) - \frac{\partial}{\partial \tau}\varphi^s_\nu(u(t-\hat{\tau}), \underline{\hat{x}}_{0,\nu}(t))\right)\tilde{\tau}(t) \\ &\triangleq \frac{\partial}{\partial \tau}\varphi^s_\nu(u(t-\hat{\tau}), \underline{\hat{x}}_{0,\nu}(t))\tilde{\tau}(t) + \tilde{\Phi}(u_0(t), \bar{\tau}(t), \hat{\tau}(t), \underline{\hat{x}}_{0,\nu}(t))\end{aligned} \tag{3.51}$$

where

$$\tilde{\Phi}(u_0(t), \bar{\tau}(t), \hat{\tau}(t), \underline{\hat{x}}_{0,\nu}(t)) = \left(\frac{\partial}{\partial \tau} \varphi_\nu^s(u(t-\bar{\tau}), \underline{\hat{x}}_{0,\nu}(t)) - \frac{\partial}{\partial \tau} \varphi_\nu^s(u(t-\hat{\tau}), \underline{\hat{x}}_{0,\nu}(t)) \right) \tilde{\tau}(t) \tag{3.52}$$

Notice that the equation (3.51) can be written as follows:

$$\begin{aligned}
\tilde{\varphi}^{s_1}(u_0(t), \hat{\tau}(t), \tau(t), \hat{x}_{0,\nu}(t)) &= \frac{\partial}{\partial u} \varphi_\nu^s(u(t-\hat{\tau}), \underline{\hat{x}}_{0,\nu}(t)) \frac{\partial}{\partial \tau} u(t-\hat{\tau}) \tilde{\tau}(t) \\
&\quad + \tilde{\Phi}(u_0(t), \bar{\tau}(t), \hat{\tau}(t), \underline{\hat{x}}_{0,\nu}(t)) \\
&= -\frac{\partial}{\partial u} \varphi_\nu^s(u(t-\hat{\tau}), \underline{\hat{x}}_{0,\nu}(t)) \dot{u}(t-\hat{\tau}) \tilde{\tau}(t) \\
&\qquad + \tilde{\Phi}(u_0(t), \bar{\tau}(t), \hat{\tau}(t), \underline{\hat{x}}_{0,\nu}(t)) \\
&= \Phi(u(t-\hat{\tau}(t)), \underline{\hat{x}}_{0,\nu}(t)) \tilde{\tau}(t) + \tilde{\Phi}(u_0(t), \bar{\tau}(t), \hat{\tau}(t), \underline{\hat{x}}_{0,\nu}(t))
\end{aligned} \tag{3.53}$$

where $\Phi(u(t-\hat{\tau}(t)), \hat{x}_0(t))$ is given by (3.15). The nonlinearity $\tilde{\varphi}^{s_1}(u_0(t), \tau(t), \hat{\tau}(t), \hat{x}_0(t))$ can be expressed as:

$$\tilde{\varphi}^{s_1}(u_0(t), \tau(t), \hat{\tau}(t), \hat{x}_0(t)) = \tilde{\varphi}^{s_1}(u_0(t), \tau(t), \hat{\tau}(t), \hat{x}_0(t)) B_\nu + \Gamma(u_0(t), \tau(t), \hat{\tau}(t), \hat{x}_0(t)), \tag{3.54}$$

where B_ν is as in (3.16) and

$$\Gamma(u_0(t), \tau(t), \hat{\tau}(t), \hat{x}_0(t)) = \varphi^{s_1}(u_0(t), \tau(t), \hat{\tau}(t), \hat{x}_0(t)) - \varphi^{s_1}(u_0(t), \tau(t), \hat{\tau}(t), \hat{x}_0(t)) B_\nu. \tag{3.55}$$

Combining (3.53),(3.54) and (3.55), the nonlinearity $\tilde{\varphi}^{s_1}$ can be expressed as

$$\begin{aligned}
\tilde{\varphi}^{s_1}(u_0(t), \tau(t), \hat{\tau}(t), \hat{x}_0(t)) &= B_\nu \Phi(u(t-\hat{\tau}(t)), \hat{x}_0(t)) \tilde{\tau}(t) + B_\nu \tilde{\Phi}(u_0(t), \bar{\tau}(t), \hat{\tau}(t), \underline{\hat{x}}_{0,\nu}(t)) \\
&\quad + \Gamma(u_0(t), \tau(t), \hat{\tau}(t), \hat{x}_0(t)).
\end{aligned} \tag{3.56}$$

Substituting $\tilde{\varphi}^{s_1}(u_0(t), \tau(t), \hat{\tau}(t), \hat{x}_0(t))$ by the above expression in the observation error equation (3.49) leads to

$$\begin{aligned}
\dot{\bar{x}}_0(t) &= \theta \left(A - S^{-1} C^T C \right) \bar{x}_0(t) + \Delta_\theta B_\nu \Phi(u(t-\hat{\tau}(t)), \hat{x}_0(t)) \tilde{\tau}(t) \\
&\quad + \Delta_\theta B_\nu \tilde{\Phi}(u_0(t), \bar{\tau}(t), \hat{\tau}(t), \underline{\hat{x}}_{0,\nu}(t)) + \Delta_\theta \Gamma(u_0(t), \tau(t), \hat{\tau}(t), \hat{x}_0(t)) \\
&\quad + \Delta_\theta \tilde{\varphi}^{s_2}(u_0(t), \tau(t), x_0(t), \hat{\tau}(t)) + \Upsilon(t) \dot{\hat{\tau}}(t).
\end{aligned} \tag{3.57}$$

From the fact that $\Delta_\theta B_\nu = \frac{1}{\theta^{\nu-1}} B_\nu$ and bearing in mind that $\tilde{\tau}(t) = \theta^\nu \bar{\tau}(t)$, the above equation can be written:

$$\begin{aligned}\dot{\bar{x}}_0(t) = \theta\left(A - S^{-1}C^TC\right)\bar{x}_0(t) + \theta B_\nu \Phi(u(t-\hat{\tau}(t)), \hat{x}_0(t))\bar{\tau}(t) \\ + \Delta_\theta B_\nu \tilde{\Phi}(u_0(t), \bar{\tau}(t), \hat{\tau}(t), \underline{\hat{x}}_{0,\nu}(t)) \\ + \Delta_\theta \Gamma(u_0(t), \tau(t), \hat{\tau}(t), \hat{x}_0(t)) + \Delta_\theta \tilde{\varphi}^{s_2}(u_0(t), \tau(t), x_0(t), \hat{\tau}(t)) + \Upsilon(t)\dot{\bar{\tau}}(t).\end{aligned} \quad (3.58)$$

Set $\bar{\eta}(t) = \bar{x}_0(t) - \Upsilon\bar{\tau}(t)$ and obtaining its time derivative, one gets:

$$\begin{aligned}\dot{\bar{\eta}}(t) = \theta(A - S^{-1}C^TC)\bar{\eta}(t) + \Delta_\theta B_\nu \tilde{\Phi}(u_0(t), \bar{\tau}(t), \hat{\tau}(t), \underline{\hat{x}}_{0,\nu}(t)) \\ + \Delta_\theta \Gamma(u_0(t), \tau(t), \hat{\tau}(t), \hat{x}_0(t)) + \Delta_\theta \tilde{\varphi}^{s_2}(u_0(t), \tau(t), x_0(t), \hat{\tau}(t)).\end{aligned} \quad (3.59)$$

Now, one proposes the following Lyapunov functions

$$\begin{aligned} V_1(\bar{\eta}(t), t) &= \bar{\eta}^T(t) S \bar{\eta}(t) \\ V_2(\bar{\tau}(t), t) &= \tfrac{\bar{\tau}^2(t)}{p(t)} \end{aligned}$$

and taking into consideration that S and $p(t)$ are given by (3.14) and (3.12), respectively.

The time derivative of $V_1(\bar{\eta}(t), t)$ is obtained using (3.58):

$$\begin{aligned}\dot{V}_1(\bar{\eta}(t)) = &-\theta V_1(\bar{\eta}(t), t) - \theta\bar{\eta}^T(t)C^TC\bar{\eta}(t) + 2\bar{\eta}^T(t)S\Delta_\theta B_\nu \tilde{\Phi}(u_0(t), \bar{\tau}(t), \hat{\tau}(t), \underline{\hat{x}}_{0,\nu}(t)) \\ &+ 2\bar{\eta}^T S \Delta_\theta \Gamma(u_0(t), \tau(t), \hat{\tau}(t), \hat{x}_0(t)) + 2\bar{\eta}^T S \Delta\tilde{\varphi}^{s_2}(u_0(t), \tau(t), x_0(t), \hat{\tau}(t)) \\ \leq &-\theta V_1(\bar{\eta}(t), t) - \theta\bar{\eta}^T(t)C^TC\bar{\eta}(t) \\ &+ 2\sqrt{\lambda_M}\sqrt{V_1(\bar{\eta}(t), t)}\left(\|\Delta_\theta B_\nu \tilde{\Phi}(u_0(t), \bar{\tau}(t), \hat{\tau}(t), \underline{\hat{x}}_{0,\nu}(t))\|\right. \\ &\left.\|\Delta_\theta \Gamma(u_0(t), \tau(t), \hat{\tau}(t), \hat{x}_0(t))\| + \|\Delta_\theta \tilde{\varphi}^{s_2}(u_0(t), \tau(t), x_0(t), \hat{\tau}(t))\|\right).\end{aligned} \quad (3.60)$$

Using the expression of $\tilde{\Phi}(u_0(t), \bar{\tau}(t), \hat{\tau}(t), \underline{\hat{x}}_{0,\nu}(t))$ given by (3.52), one gets

$$\|\Delta_\theta B_\nu \tilde{\Phi}(u_0(t), \bar{\tau}(t), \hat{\tau}(t), \underline{\hat{x}}_{0,\nu}(t))\| \leq \frac{\theta\kappa}{\sqrt{\lambda_M}}\sqrt{V_2(\bar{\tau}(t), t)}. \quad (3.61)$$

According to the definition of ν and considering the expression of $\Gamma(u_0(t), \tau(t), \hat{\tau}(t), \hat{x}_0(t))$ given by (3.55), one gets:

$$\|\Delta_\theta \Gamma(u_0(t), \tau(t), \hat{\tau}(t), \hat{x}_0(t))\| \leq \frac{1}{\sqrt{\underline{p}}}\sqrt{V_2(\bar{\tau}(t), t)}. \quad (3.62)$$

And considering the triangular structure of the function $\varphi(u(t), x(t))$, one gets:

$$\|\Delta\tilde{\varphi}^{s_2}(u_0(t), \tau(t), x_0(t), \hat{\tau}(t))\| \leq \frac{1}{\sqrt{\lambda_m}}\sqrt{V_1(\bar{\eta}(t), t)}. \quad (3.63)$$

Using the inequalities (3.61)-(3.63), inequality (3.60) leads to:

$$\dot{V}_1(\bar{\eta}(t)) \leq -\theta V_1(\bar{\eta}(t),t) - \theta(C\bar{\eta}(t))^2 + 2\sigma V_1(\bar{\eta}) + 2\kappa\theta\sqrt{V_1(\bar{\eta}(t))}\sqrt{V_2(\bar{\tau}(t),t)}$$
$$+ 2\sqrt{\frac{\lambda_M}{\underline{p}}}\sqrt{V_1(\bar{\eta}(t),t)}\sqrt{V_2(\bar{\tau}(t),t)} \tag{3.64}$$

where the conditioning number of S is given by

$$\sigma = \sqrt{\frac{\lambda_M}{\lambda_m}} \tag{3.65}$$

Now, one proceeds to obtain the time derivative of the $V_2(\bar{\tau}(t),t)$:

$$\begin{aligned}\dot{V}_2(\bar{\tau}(t)) =& 2\frac{\bar{\tau}(t)}{p(t)}\dot{\bar{\tau}}(t) - \frac{\dot{p}(t)}{p^2(t)}\bar{\tau}^2(t) \\ =& -\theta V_2(\bar{\tau}(t),t) - \theta(C\bar{\eta}(t) + C\Upsilon(t)\bar{\tau}(t))^2 + \theta(C\bar{\eta}(t))^2.\end{aligned} \tag{3.66}$$

Now, let $V(\bar{\eta}(t),\bar{\tau}(t))$ be the Lyapunov candidate function and it is defined as follows

$$V(\bar{\eta}(t),\bar{\tau}(t)) = V_1(\bar{\eta}(t),t) + V_2(\bar{\tau}(t),t) \tag{3.67}$$

According to (3.64) and (3.66), one gets:

$$\begin{aligned}\dot{V}(\bar{\eta}(t),\bar{\tau}(t),t) &= \dot{V}_1(\bar{\eta}(t)) + \dot{V}_2(\bar{\tau}(t)) \\ &\leq -\theta V(\bar{\eta}(t),\bar{\tau}(t),t) + 2\sigma V_1(\bar{\eta}(t),t) + 2\kappa\theta\sqrt{V_1(\bar{\eta}(t),t)}\sqrt{V_2(\bar{\tau}(t),t)} \\ &\quad + 2\sqrt{\frac{\lambda_M}{\underline{p}}}\sqrt{V_1(\bar{\eta}(t),t)}\sqrt{V_2(\bar{\tau}(t),t)}\end{aligned} \tag{3.68}$$

From the fact that $V_1(\bar{\eta}(t),t) \leq V(\bar{\eta}(t),\bar{\tau}(t),t)$ and $2\sqrt{V_1(\bar{\eta}(t))}\sqrt{V_2(\bar{\tau}(t),t)} \leq V_1(\bar{\eta}(t),t) + V_2(\bar{\tau}(t),t) = V(\bar{\eta}(t),\bar{\tau}(t),t)$, one gets:

$$\begin{aligned}\dot{V}(\bar{\eta}(t),\bar{\tau}(t),t) &= \dot{V}_1(\bar{\eta}(t)) + \dot{V}_2(\bar{\tau}(t),t) \\ &\leq -\left(\theta(1-\kappa) - 2\sigma - \sqrt{\frac{\lambda_M}{\underline{p}}}\right)V(\bar{\eta}(t),\bar{\tau}(t),t)\end{aligned} \tag{3.69}$$

since $\kappa(0,1)$, one chooses θ such that

$$\theta(1-\kappa) - 2\sigma - \sqrt{\frac{\lambda_M}{\underline{p}}} \geq -\frac{\theta}{2}, \tag{3.70}$$

Using the comparison lemma proposed in (Khalil and Grizzle, 2002), one gets:

$$V(\bar{\eta}(t),\bar{\tau}(t),t_0) \leq e^{-\frac{\theta}{2}(t-t_0)}V(\bar{\eta}(t_0),\bar{\tau}(t_0),t_0). \tag{3.71}$$

Coming back to $\bar{\eta}(t)$ and from the fact that $V_1 \leq V$, one gets

$$\|\bar{\eta}(t)\| \leq \sigma e^{-\frac{\theta}{2}(t-t_0)}\|\bar{\chi}(t_0)\| \tag{3.72}$$

where $\sigma = \sqrt{\frac{\lambda_M}{\lambda_m}}$ and $\bar{\chi}(t) = \begin{pmatrix} \bar{\eta}(t) \\ \tilde{\tau}(t) \end{pmatrix}$.

Now, set $\tilde{\chi}(t) = \begin{pmatrix} \tilde{\eta}(t) \\ \tilde{\tau}(t) \end{pmatrix}$ where $\bar{\eta}(t) = \Delta_\theta \tilde{\eta}(t)$. It is easy to check that

$$\|\tilde{\chi}(t)\| \leq \|\bar{\chi}(t)\| \leq \Delta_\theta \|\tilde{\chi}(t)\|, \ \forall t \geq t_0 \text{ and } \forall \theta \geq 1. \tag{3.73}$$

Additionally, one can show that $\tilde{\eta}(t) = \tilde{x}_0(t) - \Upsilon \tilde{\tau}(t)$. Therefore, one has

$$\begin{aligned} \|\tilde{\chi}(t)\| \leq& \|\tilde{\eta}(t) + \tilde{\tau}(t)\| \\ \leq& (\|\tilde{x}_0(t)\| - \|\Upsilon\|\|\tilde{\tau}(t)\|) + \|\tilde{\tau}(t)\| \\ \leq& \|\tilde{x}_0(t)\| + (1-\delta)\|\tilde{\tau}(t)\| \end{aligned} \tag{3.74}$$

Since the upper bound of $\|\Upsilon\|$ is according to Assumption 3.3.2. Now, considering (3.73) (3.74) and (3.72), one gets:

$$\|\tilde{\chi}(t)\| \leq \sigma e^{-\frac{\theta}{2}(t-t_0)} \left(\|\tilde{x}_0(t)\| + (1-\delta)\|\tilde{\tau}(t)\|\right) \tag{3.75}$$

The above proves that the error $\tilde{\chi}(t)$ that involves the observation error of states $\tilde{x}_0(t)$ and parameter $\tilde{\tau}(t)$ will converge to zero. This represents the correct estimation of the state $x_0(t)$ and the value of the unknown delay τ.

Now, we proceed to prove convergence for the subsystems in the cascade. For this purpose, one defines $\tilde{x}_j(t) = \hat{x}_j(t) - x_j(t)$ as the observation error of each predictor for $j = 1, \ldots, m$. From (3.24) and (3.28), one obtains :

$$\begin{aligned} \tilde{x}_j(t) =& e^{\bar{A}\frac{\hat{\tau}}{m}}\tilde{x}_{j-1}(t) + r_j(t) \\ &+ \int_{t-\frac{\hat{\tau}}{m}}^{t} e^{\bar{A}(t-s)} \left(\tilde{\varphi}^{(s)}(u_j(s), \hat{x}_j(t), x_j(s)) + (A-\bar{A})\tilde{x}_j(s)\right) ds \end{aligned} \tag{3.76}$$

where:

$$\tilde{\varphi}^{(s)}(u_j(s), \hat{x}_j(t), x_j(s)) = \varphi^{(s)}(u_j(s), \hat{x}_j(t)) - \varphi^{(s)}(u_j(s), x_j(t))$$

Recalling that the choice of $r_j^{'}s$, $j = 1, \ldots, m$ is given by (3.32), the following is obtained

$$\|r_j(t)\| \leq \beta e^{-\bar{a}t}\|r_j(0)\|, \ \forall t \geq 0 \tag{3.77}$$

The inequality (3.34) holds for all the predictors j.

For $j = 1$:

Taking into consideration what is shown in (3.76), the following is obtained:

$$\begin{aligned}\tilde{x}_1(t) &= e^{\bar{A}\frac{\hat{\tau}}{m}}\tilde{x}_{1-1}(t) + r_1(t) \\ &\quad + \int_{t-\frac{\hat{\tau}}{m}}^{t} e^{\bar{A}(t-s)}\left(\tilde{\varphi}^{(s)}(u_1(s), \hat{x}_1(t), x_1(s)) + (A-\bar{A})\tilde{x}_1(s)\right) ds\end{aligned}$$

In this sense $\tilde{x}_{1-1}(t)$ represents the error in the head observer, i.e.

$$\tilde{x}_{1-1}(t) = \tilde{\chi}(t)$$

Taking into consideration the above and choosing $\bar{a} \leq \frac{\theta}{2}$, one obtains the norm:

$$\|\tilde{\chi}(t)\| \leq \rho_0 e^{-\bar{a}t} \tag{3.78}$$

with:

$$\rho_0 = \sigma\left(\|\tilde{x}_0(0)\| + (1-\delta)\|\tilde{\tau}(0)\|\right) \tag{3.79}$$

Now, considering (3.34), (3.77), and (3.78), one gets

$$\begin{aligned}\|\tilde{x}_1(t)\| &\leq \beta e^{-\bar{a}\frac{\hat{\tau}}{m}}\left(\rho_0 e^{-\bar{a}t}\right) + \beta\|r_1(0)\|e^{-\bar{a}t} \\ &\quad + \int_{t-\frac{\hat{\tau}}{m}}^{t} \|e^{\bar{A}(t-s)}\|\left(\|\tilde{\varphi}^{(s)}(u_1(s), \hat{x}_1(t), x_1(s))\| + \|(A-\bar{A})\|\|\tilde{x}_1(s)\|\right) ds \\ &\leq \beta e^{-\bar{a}\frac{\hat{\tau}}{m}}\left(\rho_0 e^{-\bar{a}t}\right) + \beta\|r_1(0)\|e^{-\bar{a}t} \\ &\quad + \int_{t-\frac{\hat{\tau}}{m}}^{t} e^{-\bar{a}(t-s)}\left(\left(L_\varphi + \|(A-\bar{A})\|\right)\|\tilde{x}_1(s)\|\right) ds\end{aligned} \tag{3.80}$$

The above can be rewritten as follows

$$\|\tilde{x}_1(t)\| \leq \beta e^{-\bar{a}\frac{\hat{\tau}}{m}}\left(\rho_0 e^{-\bar{a}t}\right) + \beta\|r_1(0)\|e^{-\bar{a}t} + \int_{t-\frac{\hat{\tau}}{m}}^{t} e^{-\bar{a}(t-s)}\left(\varrho\|\tilde{x}_1(s)\|\right) ds \tag{3.81}$$

where

$$\varrho = L_\varphi + \|(A-\bar{A})\|. \tag{3.82}$$

One can express the above as follows

$$\begin{aligned}\|\tilde{x}_1(t)\| &\leq \beta\left(e^{-\bar{a}\frac{\hat{\tau}}{m}}\left(\rho_0 e^{-\bar{a}t}\right) + e^{\bar{a}t}\|r_1(0)\|\right) + \int_{t-\frac{\hat{\tau}}{m}}^{t} e^{-\bar{a}(t-s)}\varrho\|\tilde{x}_1(s)\|ds \\ &\leq \beta\left(\rho_0 e^{-\bar{a}\frac{\hat{\tau}}{m}} + \|r_1(0)\|\right)e^{-\bar{a}t} + \varrho\int_{t-\frac{\hat{\tau}}{m}}^{t} e^{-\bar{a}(t-s)}\|\tilde{x}_1(s)\|ds\end{aligned} \tag{3.83}$$

Let us define $z_1(t) = e^{\bar{a}t}\|\tilde{x}_1(t)\|$ and

$$K_1(t) = \beta\left(\rho_0 e^{-\bar{a}\frac{\hat{\tau}}{m}} + \|r_1(0)\|\right) \tag{3.84}$$

Bearing in mind the above, the equation (3.83) can be rewritten as follows

$$z_1(t) \leq K_1(t) + \varrho\int_{t-\frac{\hat{\tau}}{m}}^{t} z_1(s)ds \tag{3.85}$$

Considering the proposal in equation , the proposal in lemma B.2.1 is fulfilled, obtaining

$$\begin{aligned} z_1(t) \leq& \frac{1}{1-\varrho\frac{\hat{\tau}}{m}}\left(K_1(t) + \varrho\int_{\frac{\hat{\tau}}{m}}^{0} z_1(s)ds\right) \\ =& \frac{1}{1-\varrho\frac{\hat{\tau}}{m}}\left(K_1(t) + \varrho\int_{\frac{\hat{\tau}}{m}}^{0} e^{\bar{a}s}\|\tilde{x}_1(s)\|ds\right) \\ \leq& \frac{1}{1-\varrho\frac{\hat{\tau}}{m}}\left(K_1(t) + \varrho\int_{\frac{\hat{\tau}}{m}}^{0} \|\tilde{x}_1(s)\|ds\right) \end{aligned} \tag{3.86}$$

Coming back to $\|\tilde{x}_1(t)\|$ and according to (3.86), yields to

$$\begin{aligned} \|\tilde{x}_1(t)\| \leq& \frac{1}{1-\varrho\frac{\hat{\tau}}{m}}\left(\beta\left(\rho_0 e^{-\bar{a}\frac{\hat{\tau}}{m}} + \|r_1(0)\|\right) + \varrho\int_{\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_1(s)\|ds\right)e^{-\bar{a}t} \\ \triangleq& \rho_1 e^{-\bar{a}t} \end{aligned} \tag{3.87}$$

with

$$\rho_1 = \frac{1}{1-\varrho\frac{\hat{\tau}}{m}}\left(\beta\left(\rho_0 e^{-\bar{a}\frac{\hat{\tau}}{m}} + \|r_1(0)\|\right) + \varrho\int_{\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_1(s)\|ds\right) \tag{3.88}$$

Subsequently, the expression ρ_0 is substituted in ρ_1, obtaining

$$\begin{aligned} \rho_1 =& \frac{1}{1-\varrho\frac{\hat{\tau}}{m}}\left(\beta\sigma\left(\|\tilde{x}_0(0)\| + (1-\delta)\|\tilde{\tau}(0)\|\right)e^{-\bar{a}\frac{\hat{\tau}}{m}} + \beta\|r_1(0)\| + \varrho\int_{\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_1(s)\|ds\right) \\ =& \beta\zeta_m\sigma\left(\|\tilde{x}_0(0)\| + (1-\delta)\|\tilde{\tau}(0)\|\right) + \frac{\beta}{1-\varrho\frac{\hat{\tau}}{m}}\|r_1(0)\| + \frac{\varrho}{1-\varrho\frac{\hat{\tau}}{m}}\int_{\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_1(s)\|ds \end{aligned} \tag{3.89}$$

with

$$\zeta_m = \frac{e^{-\bar{a}\frac{\hat{\tau}}{m}}}{1-\varrho\frac{\hat{\tau}}{m}}. \tag{3.90}$$

Following this line, it is proposed that for all predictors $j = 1, \ldots, j-1$, the error is shown as follows:

$$\|\tilde{x}_i(t)\| \leq \rho_i e^{-\bar{a}t},\ \forall t \geq 0 \tag{3.91}$$

It has been shown early on that the first predictor of the cascade converges to zero (3.87), now we proceed to show for $j = 2$. Let us define the following auxiliary function:

$$\vartheta_i(t) = \int_{t-\frac{\hat{\tau}}{m}}^{t} e^{\bar{A}(t-s)} \left(\tilde{\varphi}^{(s)}(u_i(s), \hat{x}_i(s), x_i(s)) + (A - \bar{A})\tilde{x}_i(s)\right) ds$$

Now, according to equation (3.76) and obtaining its integration, one gets:

$$\begin{aligned}\tilde{x}_j(t) =& e^{\bar{A}j\frac{\hat{\tau}}{m}}\tilde{x}_{1-1}(t) + \sum_{i=1}^{j}\left(e^{\bar{A}(j-i)\frac{\hat{\tau}}{m}}r_i(t)\right) + \sum_{i=1}^{j}\left(e^{\bar{A}(j-i)\frac{\hat{\tau}}{m}}\vartheta_i(t)\right)\\ =& e^{\bar{A}j\frac{\hat{\tau}}{m}}\tilde{\chi}(t) + \sum_{i=1}^{j}\left(e^{\bar{A}(j-i)\frac{\hat{\tau}}{m}}r_i(t)\right) + \sum_{i=1}^{j}\left(e^{\bar{A}(j-i)\frac{\hat{\tau}}{m}}\vartheta_i(t)\right) \qquad (3.92)\end{aligned}$$

Now, obtaining the norm of the above expression, one gets

$$\begin{aligned}\|\tilde{x}_j(t)\| \leq& \|e^{\bar{A}j\frac{\hat{\tau}}{m}}\|\|\tilde{\chi}(t)\| + \sum_{i=1}^{j}\left(\|e^{\bar{A}(j-i)\frac{\hat{\tau}}{m}}\|\|r_i(0)\|\right) + \sum_{i=1}^{j}\left(\|e^{\bar{A}(j-i)\frac{\hat{\tau}}{m}}\|\|\vartheta_i(t)\|\right)\\ \leq& \beta e^{-\bar{a}j\frac{\hat{\tau}}{m}}\|\tilde{\chi}(t)\| + \beta\left(\sum_{i=1}^{j} e^{-\bar{a}(j-i)\frac{\hat{\tau}}{m}}\|r_i(0)\|\right)e^{-\bar{a}t} + \sum_{i=1}^{j}\|e^{\bar{A}(j-i)\frac{\hat{\tau}}{m}}\vartheta_i(t)\|. \qquad (3.93)\end{aligned}$$

Focusing on term $\|e^{\bar{A}(j-i)\frac{\tau}{m}}\vartheta_i(t)\|$, one proceeds to obtain its norm as follows

$$\begin{aligned}\|e^{\bar{A}(j-i)\frac{\tau}{m}}\vartheta_i(t)\| \leq& e^{-\bar{a}(j-i)\frac{\tau}{m}}\int_{t-\tau/m}^{t} e^{-\bar{a}(t-s)}\left(\varrho\|\tilde{x}_i(s)\|\right) ds\\ \leq& e^{-\bar{a}(j-i)\frac{\tau}{m}}\left(\varrho\int_{t-\tau/m}^{t} e^{-\bar{a}(t-s)}\|\tilde{x}_i(s)\|ds\right) \qquad (3.94)\end{aligned}$$

with ϱ given by (3.82).

Substituting Eq.(3.94) in Eq.(3.93), one obtains:

$$\|\tilde{x}_j(t)\| \leq \beta e^{-\bar{a}j\frac{\hat{\tau}}{m}}\|\tilde{\chi}(t)\| + \beta\left(\sum_{i=1}^{j} e^{-\bar{a}(j-i)\frac{\hat{\tau}}{m}}\|r_i(0)\|\right)e^{-\bar{a}t} \qquad (3.95)$$

the above equation can be rewritten as follows

$$\|\tilde{x}_j(t)\| \leq \beta e^{-\bar{a}j\frac{\hat{\tau}}{m}}\|\tilde{\chi}(t)\| + \beta\left(\sum_{i=1}^{j} e^{-\bar{a}(j-i)\frac{\tau}{m}}\|r_i(0)\|\right)e^{-\bar{a}t} + \varrho\int_{t-\frac{\hat{\tau}}{m}}^{t} e^{-\bar{a}(t-s)}\|\tilde{x}_j(s)\|ds. \qquad (3.96)$$

Obtaining the norm of the term $\int_{t-\frac{\hat{\tau}}{m}}^{t} e^{-\bar{a}(t-s)}\|\tilde{x}_i(s)\|ds$ using Eq. (3.91), one gets:

$$\begin{aligned}
\int_{t-\frac{\hat{\tau}}{m}}^{t} e^{-\bar{a}(t-s)}\|\tilde{x}_i(s)\|ds &\leq \int_{t-\frac{\hat{\tau}}{m}}^{t} e^{-\bar{a}(t-s)}\left(\rho_i e^{-\bar{a}s}\right) ds \\
&\leq \rho_i \int_{t-\frac{\hat{\tau}}{m}}^{t} e^{-\bar{a}(t-s)-\bar{a}s} \\
&\leq \rho_i \int_{t-\frac{\hat{\tau}}{m}}^{t} e^{-\bar{a}t} \\
&\leq \rho_i \frac{\hat{\tau}}{m} e^{-\bar{a}t} \\
\int_{t-\frac{\hat{\tau}}{m}}^{t} e^{-\bar{a}(t-s)}\|\tilde{x}_i(s)\|ds &\leq \rho_i \frac{\hat{\tau}}{m} e^{-\bar{a}t} \qquad (3.97)
\end{aligned}$$

Using the equations (3.34), (3.75) and (3.97), the inequality (3.96) results in the following form:

$$\begin{aligned}
\|\tilde{x}_j(t)\| \leq \beta e^{\bar{a}j\frac{\hat{\tau}}{m}}\rho_0 e^{-\bar{a}t} &+ \varrho\frac{\hat{\tau}}{m}\sum_{i=1}^{j-1} e^{-\bar{a}(j-i)\frac{\tau}{m}}\left(\rho_i e^{-\bar{a}t}\right) \\
&+ \varrho\int_{t-\frac{\hat{\tau}}{m}}^{t} e^{-\bar{a}(t-s)}\|\tilde{x}_j(s)\|ds + \beta\left(\sum_{i=1}^{j} e^{-\bar{a}(j-i)\frac{\hat{\tau}}{m}}\|r_i(0)\|\right)e^{-\bar{a}t}
\end{aligned}$$

Rewritten the above:

$$\begin{aligned}
\|\tilde{x}_j(t)\| &\leq \beta\rho_0 e^{-\bar{a}j\frac{\hat{\tau}}{m}} e^{-at} + \beta\|r_j(0)\|e^{-\bar{a}t} + \beta\left(\sum_{i=1}^{j-1} e^{-\bar{a}(j-i)\frac{\hat{\tau}}{m}}\|r_i(0)\|\right)e^{-\bar{a}t} \\
&\quad + \varrho\int_{t-\frac{\hat{\tau}}{m}}^{t} e^{-\bar{a}(t-s)}\|\tilde{x}_j(s)\|ds + \varrho\sum_{i=1}^{j-1}\rho_i\frac{\hat{\tau}}{m}e^{-\bar{a}t} \\
\|\tilde{x}_j(t)\| &\leq \left(\beta\left(\rho_0 e^{-\bar{a}\frac{\hat{\tau}}{m}(j)} + \|r_j(0)\|\right) + \sum_{i=1}^{j-1} e^{-\bar{a}\frac{\hat{\tau}}{m}(j-i)}\left(\varrho\rho_i\frac{\hat{\tau}}{m} + \beta\|r_i(0)\|\right)\right)e^{-\bar{a}t} \\
&\quad + \varrho\int_{t-\frac{\hat{\tau}}{m}}^{t} e^{-\bar{a}(t-s)}\|\tilde{x}_j(s)\|ds \qquad (3.98)
\end{aligned}$$

Now, one considers $z_j(t) = e^{\bar{a}t}\|\tilde{x}_j(t)\|$, thus:

$$K_j(t) = \beta\left(\rho_0 e^{-\bar{a}\frac{\hat{\tau}}{m}(j)} + \|r_j(0)\|\right) + \sum_{i=1}^{j-1} e^{-\bar{a}\frac{\hat{\tau}}{m}(j-i)}\left(\varrho\rho_i\frac{\hat{\tau}}{m} + \beta\|r_i(0)\|\right) \qquad (3.99)$$

Therefore the equation (3.98) yields to:

$$z_j(t) \leq K_j(t) + \varrho\int_{t-\frac{\hat{\tau}}{m}}^{t} z_j(s)ds \qquad (3.100)$$

Considering that equation (3.38) is fulfilled and the lemma B.2.1 is satisfied, one obtains:

$$
\begin{aligned}
z_j(t) &\le \frac{1}{1-\varrho\frac{\hat{\tau}}{m}}\left(K_j(t)+\varrho\int_{-\frac{\hat{\tau}}{m}}^{0} z_j(s)ds\right) \\
&\le \frac{1}{1-\varrho\frac{\hat{\tau}}{m}}\left(K_j(t)+\varrho\int_{-\frac{\hat{\tau}}{m}}^{0} e^{-\bar{a}s}\|x_j(s)\|ds\right) \\
z_j(t) &\le \frac{1}{1-\varrho\frac{\hat{\tau}}{m}}\left(K_j(t)+\varrho\int_{-\frac{\hat{\tau}}{m}}^{0} \|\tilde{x}_j(s)\|ds\right). \qquad (3.101)
\end{aligned}
$$

Coming back to terms $\|\tilde{x}_j(t)\|$ and considering the above equation, one gets:

$$
\begin{aligned}
\|\tilde{x}_j(t)\| \le \frac{1}{1-\varrho\frac{\hat{\tau}}{m}} &\left(\beta\left(\rho_0 e^{-\bar{a}\frac{\hat{\tau}}{m}(j)}+\|r_j(0)\|\right)+\sum_{i=1}^{j-1} e^{-\bar{a}\frac{\hat{\tau}}{m}(j-i)}\left(\varrho\rho_i\frac{\hat{\tau}}{m}+\beta\|r_i(0)\|\right)\right. \\
&\left. +\varrho\int_{-\tau/m}^{0}\|\tilde{x}_j(s)\|ds\right) e^{-\bar{\alpha}t} \qquad (3.102)
\end{aligned}
$$

Combining terms, we obtain:

$$
\begin{aligned}
\rho_j = \frac{1}{1-\varrho\frac{\hat{\tau}}{m}} &\left(\beta\left(\rho_0 e^{-\bar{a}\frac{\hat{\tau}}{m}(j)}+\|r_j(0)\|\right)+\sum_{i=1}^{j-1} e^{-\bar{a}\frac{\hat{\tau}}{m}(j-i)}\left(\varrho\rho_i\frac{\hat{\tau}}{m}+\beta\|r_i(0)\|\right)\right. \\
&\left. +\varrho\int_{-\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_j(s)\|ds\right) \qquad (3.103)
\end{aligned}
$$

By rewriting the equation (3.102), one gets:

$$
\|\tilde{x}_j(t)\| \le \rho_j e^{-\bar{a}t} \qquad (3.104)
$$

Taking into consideration the equation (3.103), one gets:

$$
\begin{aligned}
\left(1-\varrho\frac{\hat{\tau}}{m}\right)\rho_j =& \beta\left(e^{-\bar{a}\frac{\hat{\tau}}{m}(j)}\rho_0+\|r_j(0)\|\right)+\sum_{i=1}^{j-1} e^{-\bar{a}\frac{\hat{\tau}}{m}(j-i)}\left(\varrho\rho_i\frac{\hat{\tau}}{m}+\beta\|r_i(0)\|\right) \\
&+\varrho\int_{-\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_j(s)\|ds \qquad (3.105)
\end{aligned}
$$

This can also be written in the following way:

$$
\begin{aligned}
\rho_j &= \beta e^{-\bar{a}\frac{\hat{\tau}}{m}(j)}\rho_0+\sum_{i=1}^{j} e^{-\bar{a}\frac{\hat{\tau}}{m}(j-i)}\left(\varrho\frac{\hat{\tau}}{m}\rho_i+\beta\|r_i(0)\|\right)+\varrho\int_{-\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_j(s)\|ds \\
&= \beta e^{-\bar{a}\frac{\hat{\tau}}{m}(j)}\rho_0+\varrho\frac{\hat{\tau}}{m}\rho_j+\beta\|r_j(0)\|+\sum_{i=1}^{j-1} e^{-\bar{a}\frac{\hat{\tau}}{m}(j-i)}\left(\varrho\frac{\hat{\tau}}{m}\rho_i+\beta\|r_i(0)\|\right) \\
&\quad +\varrho\int_{-\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_j(s)\|ds. \qquad (3.106)
\end{aligned}
$$

Now, equation (3.106) is considered in a range of $j-1$, one gets:

$$\rho_{j-1} = \beta\left(e^{-\bar{a}\frac{\hat{\tau}}{m}(j-1)}\rho_0\right) + \sum_{i=1}^{j-1} e^{-\bar{a}\frac{\hat{\tau}}{m}(j-1-i)}\left(\varrho\frac{\hat{\tau}}{m}\rho_i + \beta\|r_i(0)\|\right) + \varrho\int_{-\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_{j-1}(s)\|ds$$
$$= e^{\bar{a}\frac{\hat{\tau}}{m}}\left(\beta\left(e^{-\bar{a}\frac{\hat{\tau}}{m}(j)}\rho_0\right) + \sum_{i=1}^{j-1} e^{-\bar{a}\frac{\hat{\tau}}{m}(j-i)}\left(\varrho\frac{\hat{\tau}}{m}\rho_i + \beta\|r_i(0)\|\right) + e^{-\bar{a}\frac{\hat{\tau}}{m}}\varrho\int_{-\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_{j-1}(s)\|ds\right) \tag{3.107}$$

Subtracting equation (3.107) from equation (3.103), we obtain:

$$\rho_j - \rho_{j-1}e^{-\bar{a}\frac{\hat{\tau}}{m}} = \varrho\frac{\hat{\tau}}{m}\rho_j + \beta\|r_j(0)\| + \varrho\left(\int_{-\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_j(s)\|ds - e^{-\bar{a}\frac{\tau}{m}}\int_{-\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_{j-1}(s)\|ds\right) \tag{3.108}$$

Taking ρ_j out of the previous equation, we have:

$$\rho_j - \varrho\frac{\hat{\tau}}{m}\rho_j = \rho_{j-1}e^{-\bar{a}\frac{\hat{\tau}}{m}} + \beta\|r_j(0)\| + \varrho\left(\int_{-\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_j(s)\|ds - e^{-\bar{a}\frac{\hat{\tau}}{m}}\int_{-\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_{j-1}(s)\|ds\right)$$
$$\rho_j = \frac{e^{-\bar{a}\frac{\hat{\tau}}{m}}}{1-\varrho\frac{\hat{\tau}}{m}}\rho_{j-1} + \frac{\beta\|r_j(0)\|}{1-\varrho\frac{\hat{\tau}}{m}}$$
$$+ \frac{\varrho}{1-\varrho\frac{\hat{\tau}}{m}}\left(\int_{-\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_j(s)\|ds - e^{-\bar{a}\frac{\hat{\tau}}{m}}\int_{-\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_{j-1}(s)\|ds\right). \tag{3.109}$$

By grouping terms and considering that the recursive equation is valid for $j \geq 2$:

$$\left(\rho_j - \frac{\varrho}{1-\varrho\frac{\hat{\tau}}{m}}\int_{-\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_j(s)\|ds\right) = \frac{e^{-\bar{a}\frac{\hat{\tau}}{m}(j-1)}}{\left(1-\varrho\frac{\hat{\tau}}{m}\right)^{j-1}}\left(\rho_1 - \frac{\varrho}{1-\varrho\frac{\hat{\tau}}{m}}\int_{-\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_1(s)\|ds\right)$$
$$+ \sum_{i=2}^{j}\frac{e^{-\bar{a}\frac{\hat{\tau}}{m}(j-i)}}{\left(1-\varrho\frac{\hat{\tau}}{m}\right)^{j-i}}\frac{\beta\|r_i(0)\|}{1-\varrho\frac{\hat{\tau}}{m}}$$
$$= \frac{e^{-\bar{a}\frac{\hat{\tau}}{m}(j-1)}}{\left(1-\varrho\frac{\hat{\tau}}{m}\right)^{j-1}}\left(\rho_1 - \frac{\varrho}{1-\varrho\frac{\hat{\tau}}{m}}\int_{-\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_1(s)\|ds\right)$$
$$+ \sum_{i=2}^{j}\frac{e^{-\bar{a}\frac{\hat{\tau}}{m}(j-i)}}{\left(1-\varrho\frac{\hat{\tau}}{m}\right)^{j-i}}\frac{\beta\|r_i(0)\|}{1-\varrho\frac{\tau}{m}}. \tag{3.110}$$

Considering ρ_1 given by equation (3.89), one has:

$$\left(\rho_1 - \frac{\varrho}{1-\varrho\frac{\hat{\tau}}{m}}\int_{-\frac{\hat{\tau}}{m}}^{0}\|\tilde{x}_1(s)\|ds\right) = \beta\frac{e^{-\bar{a}\frac{\hat{\tau}}{m}}}{1-\varrho\frac{\hat{\tau}}{m}}\rho_0 + \frac{\beta}{1-\varrho\frac{\hat{\tau}}{m}}\|r_1(0)\| \tag{3.111}$$

and substituting the above in equation (3.110) , one gets:

$$\rho_j - \frac{\varrho}{1-\varrho\frac{\hat{\tau}}{m}} \int_{-\frac{\hat{\tau}}{m}}^{0} \|\tilde{x}_j(s)\| ds = \frac{e^{-\bar{a}\frac{\hat{\tau}}{m}(j)}}{\left(1-\varrho\frac{\hat{\tau}}{m}\right)^j} \beta\rho_0 + \frac{e^{-\bar{a}\frac{\hat{\tau}}{m}(j-1)}}{\left(1-\varrho\frac{\hat{\tau}}{m}\right)^j} \beta\|r_1(0)\| + \sum_{i=2}^{j} \frac{e^{-\bar{a}\frac{\hat{\tau}}{m}(j-i)}}{\left(1-\varrho\frac{\hat{\tau}}{m}\right)^{j+1-i}} \beta\|r_i(0)\| \tag{3.112}$$

Rewriting yields the following:

$$\begin{aligned}
\rho_j &= \frac{\varrho}{1-\varrho\frac{\hat{\tau}}{m}} \int_{-\frac{\hat{\tau}}{m}}^{0} \|\tilde{x}_j(s)\| ds + \frac{e^{-\bar{a}\frac{\hat{\tau}}{m}(j)}}{\left(1-\varrho\frac{\hat{\tau}}{m}\right)^j} \beta\rho_0 + \sum_{i=1}^{j} \frac{e^{-\bar{a}\frac{\hat{\tau}}{m}(j-i)}}{\left(1-\varrho\frac{\hat{\tau}}{m}\right)^{j+1-i}} \beta\|r_i(0)\| \\
&= \frac{\varrho}{1-\varrho\frac{\hat{\tau}}{m}} \int_{-\frac{\hat{\tau}}{m}}^{0} \|\tilde{x}_j(s)\| ds + \beta\frac{e^{-\bar{a}\frac{\tau}{m}(j)}}{\left(1-\varrho\frac{\hat{\tau}}{m}\right)^j} \rho_0 + \sum_{i=1}^{j} \frac{e^{-\bar{a}\frac{\hat{\tau}}{m}(j-i)}}{\left(1-\varrho\frac{\hat{\tau}}{m}\right)^{j+1-i}} \beta\|r_i(0)\| \\
&= \frac{\varrho}{1-\varrho\frac{\hat{\tau}}{m}} \int_{-\frac{\hat{\tau}}{m}}^{0} \|\tilde{x}_j(s)\| ds + \beta\zeta_m^j\rho_0 + \frac{\beta}{1-\varrho\frac{\hat{\tau}}{m}} \sum_{k=0}^{j-1} \zeta_m^k \|r_{j-k}(0)\|
\end{aligned}$$

$$\rho_j = \frac{\varrho}{1-\varrho\frac{\hat{\tau}}{m}} \int_{-\frac{\hat{\tau}}{m}}^{0} \|\tilde{x}_j(s)\| ds + \beta\zeta_m^j\sigma\left(\|\tilde{x}_0(0)\| + (1-\delta)\|\tilde{\tau}(0)\|\right) + \frac{\beta}{1-\varrho\frac{\hat{\tau}}{m}} \sum_{k=0}^{j-1} \zeta_m^k \|r_{j-k}(0)\| \tag{3.113}$$

This results in ρ_j given by equation (3.41). This ends the proof of Theorem 3.4.1. ■

3.5 Illustrative examples

This section shows the effectiveness of the proposed cascade observer to jointly estimate the delay-free states and the value of the unknown delay present in a nonlinear system. The application of this observer is shown in two examples; an academic system and a robotic arm system.

3.5.1 Academic example

Considering the following system proposed in Farza et al. (2015a):

$$\mathcal{SYS}:\begin{cases}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}x_2(t)\\ -x_1(t)+0.5\tanh\left(x_1(t)x_2\right)+x_1(t)u(t)\end{bmatrix}\\ \quad y_\tau(t) = Cx(t-\tau)=x_1(t-\tau)\end{cases} \tag{3.114}$$

where $x(t)=[x_1(t),\ x_2(t)]^T\in\mathbb{R}^2$ denotes the actual state vector; $y_\tau(t)\in\mathbb{R}^p$ with $p=1$ denotes the available output; the system input is denoted by $u(t)\in\mathbb{R}$. It is important to emphasize that system (3.114) presents in the available output an unknown time delay $\tau>0$.

The above system can be rewritten as follows:

$$\mathcal{SYS}:\begin{cases}\dot{x}(t) = & Ax(t)+\varphi(u(t),x(t))\\ y_\tau(t) = & Cx(t-\tau)=x_1(t-\tau)\end{cases} \tag{3.115}$$

with:

$$\begin{aligned}A&=\begin{bmatrix}0&1\\0&0\end{bmatrix},\ \varphi(u(t),x(t))=\begin{pmatrix}0\\ -x_1(t)+0.5\tanh\left(x_1(t)x_2(t)\right)+x_1(t)u(t)\end{pmatrix},\\ C&=\begin{bmatrix}1&0\end{bmatrix},\end{aligned} \tag{3.116}$$

and the system input $u(t)$ is given by the following form:

$$u(t)=0.1\sin(0.2\pi t). \tag{3.117}$$

One can see that system (3.115) is the same as shown in (3.1) and presents unknown delay at the output. The challenge is to estimate together the current state vector of system

(3.115) and the value of the unknown delay. For this, the observer is designed as follows:

$$\left\{\begin{aligned}
&\text{For } j = && 0 \\
&\dot{\hat{x}}_0(t) && = A\hat{x}_0(t) + \varphi^s(u(t-\hat{\tau}), \hat{x}_0(t)) - G_0(t), \\
&G_0(t) && = \theta\Delta_\theta^{-1}\left(S^{-1} + p(t)\Upsilon(t)\Upsilon^T(t)\right)C^T(C\hat{x}_0(t) - y_\tau(t)), \\
&\dot{\hat{\tau}}(t) && = -\theta^{\nu+1}p(t)\Upsilon^T(t)C^T(C\hat{x}_0(t) - y_\tau(t)), \\
&\dot{\Upsilon}(t) && = \theta\left(A - S^{-1}C^TC\right)\Upsilon(t) + \theta B_\nu\Phi(u(t-\hat{\tau}), \hat{x}_0(t)), \\
&\dot{p}(t) && = -\theta p^2(t)\Upsilon^T(t)C^TC\Upsilon(t) + \theta p(t) \\
&\text{For } j = && 1, \ldots, m \\
&\dot{\hat{x}}_j(t) && = A\hat{x}_j(t) + \varphi^{(s)}(u_j(t), \hat{x}_j(t)) - G_j(t), \\
&G_j(t) && = e^{\bar{A}\frac{\hat{\tau}}{m}}\Big(G_{j-1}(t) + (A-\bar{A})\left(\hat{x}_j(t-\tfrac{\hat{\tau}}{m}) - \hat{x}_{j-1}(t)\right) \\
& && \qquad +\varphi^{(s)}(u_{j-1}(t), \hat{x}_j(t-\tfrac{\hat{\tau}}{m})) - \varphi^{(s)}(u_{j-1}(t), \hat{x}_{j-1}(t))\Big)
\end{aligned}\right. \tag{3.118}$$

Before proceedings, it is necessary to obtain some terms that are involved in the head of the observer ($j = 0$). According to Assumption 3.3.1 and equation (3.11), one obtains $\nu = 2$. Since in the vector of nonlinear functions $\varphi(u(t), x(t))$ of system (3.114), the input exists only in the second state, i.e. $k = 2$.

Now, considering $x_0(t) = x(t-\tau)$, the nonlinear vector function can be rewritten as follows:

$$\varphi(\hat{x}_0(t), u(t-\hat{\tau})) = \begin{pmatrix} 0 \\ -x_{1,0}(t) + 0.5\tanh\left(x_{1,0}(t)x_{2,0}(t)\right) + x_{1,0}(t)u(t-\hat{\tau}) \end{pmatrix} \tag{3.119}$$

Bearing in mind the above, one obtains $\Phi(u(t-\hat{\tau}), \hat{z})$ as follows:

$$\begin{aligned}
\Phi(u(t-\hat{\tau}), \hat{x}_0(t)) &= -\frac{\partial}{\partial u}\varphi^s(u(t-\hat{\tau}(t)), \underline{\hat{x}_{0,\nu}})\dot{u}(t-\hat{\tau}(t)), \\
&= -\,0.02\pi\hat{x}_{1,0}(t)\cos(0.2\pi(t-\hat{\tau}(t)))
\end{aligned} \tag{3.120}$$

and according to equation (3.11) the column vector B_ν is

$$B_\nu(\nu) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3.121}$$

Three sets of simulations are presented below to demonstrate different scenarios that the observer can deal with. All simulations will have the following initial conditions and parameters. Note that all simulations were obtained without the saturation of nonlinearities and state. Since one considers that states are bounded with the initial conditions.

The matrix S is the unique solution to the equation (3.14) and Δ_θ are as follows:

$$S = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \ \Delta_\theta = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\theta} \end{bmatrix}, \tag{3.122}$$

and the tuning parameter is set to $\theta = 1.2$. The initial conditions for the original system are:

$$x(0) = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} \tag{3.123}$$

for the head observer ($j = 0$):

$$\begin{aligned} &\hat{x}(s) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ \forall s \in [-\hat{\tau}(0), 0], \Upsilon(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\hat{\tau}(0) = 0, \ p(0) = 1, \end{aligned} \tag{3.124}$$

and for the predictors ($j = 1, \ldots, m$):

$$\hat{x}_j(s) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ \forall s \in [-\hat{\tau}(0), 0], \ \bar{A} = A - \lambda I_{2\times 2}, \lambda = 1. \tag{3.125}$$

The first and second set of simulations are considering that the number of predictors is equal to $m = 5$. In the first simulation, the available output presents a time delay equal to $\tau = 0.5$ sec. Figures 3.1-3.7 show the results of the first simulation. The first $(x_1(t-\tau))$ and second $(x_2(t-\tau))$ delayed states estimated by the adaptive observer at the head of the cascade are shown in Figures 3.1-3.2, respectively. Similarly, this observer achieves the estimation of the delay value present at the output of the system. Figure 3.3 shows the estimation of the unknown delay by the observer in the head of the observer. It is evident that the observer is able to jointly estimate the states and the delay present in the system output. Theorem 3.4.1 shows that the observation error of the states and the parameter converges exponentially to zero. This is shown in Figure 3.4, where the norm of the observation error and the unknown delay are shown. The actual state vector estimated by the last predictor $x_m(t)$, $m = 5$ in the cascade is shown in Figures 3.5 and 3.6. Figure 3.5 shows the estimation of the actual first $x_1(t)$ and Figure 3.6 shows the estimation of the actual second state $x_2(t)$. The norm of the state observation error is shown in Figure 3.7. These figures show that the observer (3.118) achieves the joint estimation of the actual state of the system even in the presence of unknown time-delay in the output signal.

The second simulation evaluates the performance of the observer in a more demanding case of output delay. For this simulation, the output delay is considered to be $\tau = 5$ sec.

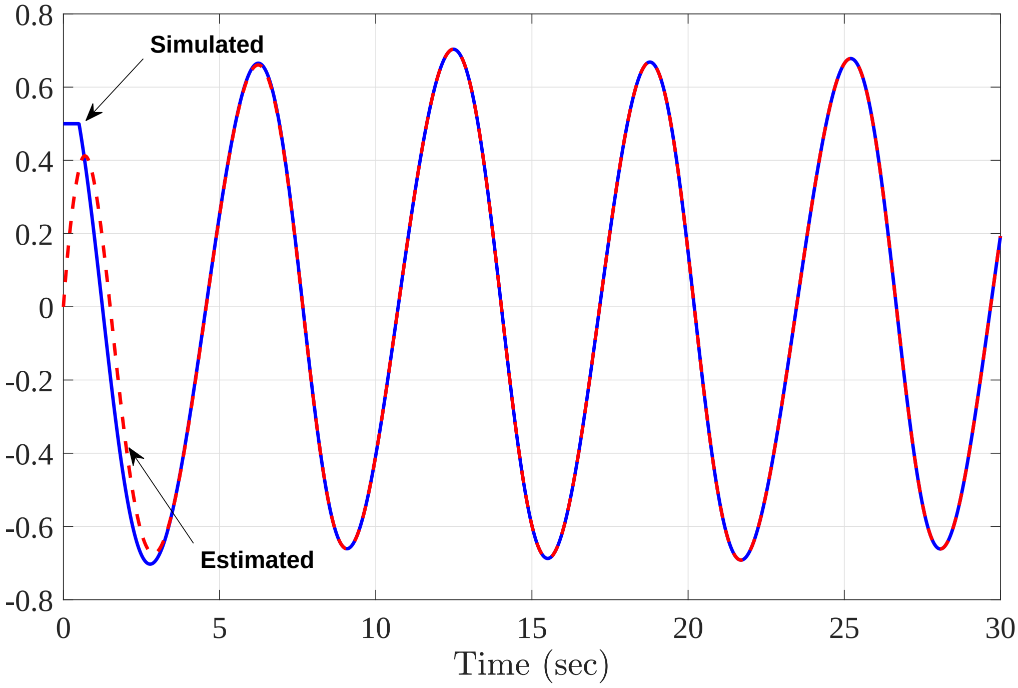

Figure 3.1: First set of simulations: Estimation of the first delayed state $x_1(t-\tau)$

The results of the second set of simulations are shown in Figures 3.8-3.14. The delayed states $x_1(t-\tau)$ and $x_2(t-\tau)$ are shown in Figures 3.8 and 3.9, respectively. The delay estimation is shown in Figure 3.10. It is evident that the observer in the head manages to estimate the delayed states and the delay value, even when the value is increased. The error norm of states and delay is shown in Figure 3.11. The actual first state $x_1(t)$ estimation is shown in Figure 3.12 and the actual second state $x_2(t)$ is shown in Figure 3.13. Recall that these are estimated by the last predictor $x_m(t)$, $m = 5$ in the cascade. The norm of the state observation error is shown in Figure 3.14.

The third simulation focuses on evaluating how the number of predictors affects the convergence of the system. For this purpose, the delay present at the output is considered to be equal to $\tau = 5$ sec. The observer design of the form (3.118) is considered, taking into account $m = 5$, $m = 8$, and $m = 20$. For the purpose of a clear visualization of the phenomenon caused by the selection of the number of predictors, Figure 3.15 shows a comparison of the error norm of these observers. If we compare the predictors $m = 20$ with respect to $m = 5$, it is shown that with a larger number of predictors a smoother convergence is obtained. In all cases, the error norm converges to zero.

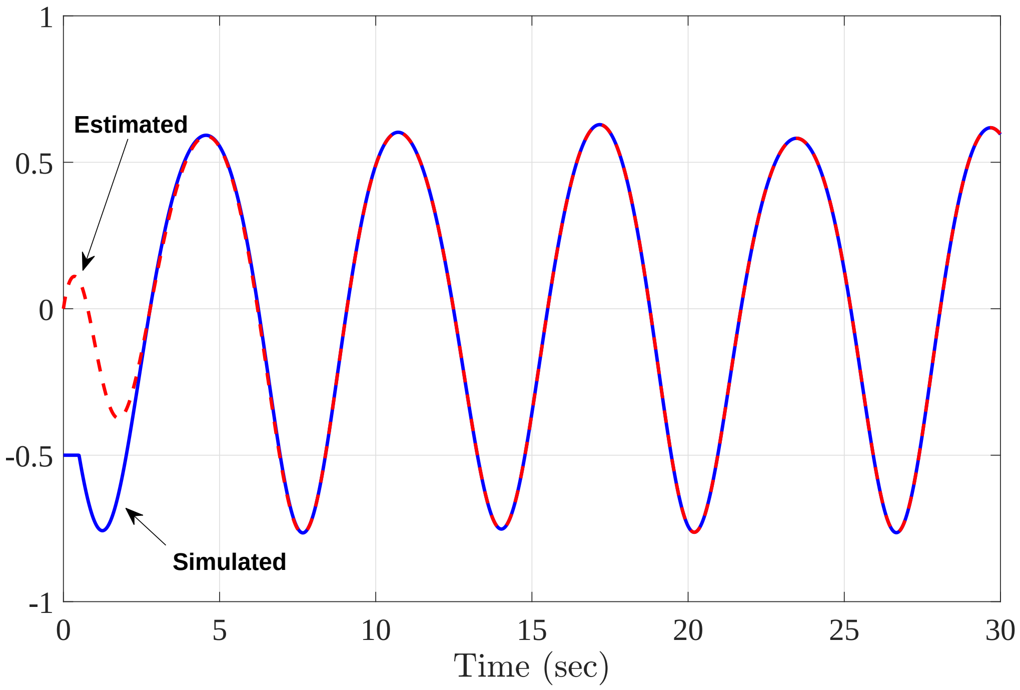

Figure 3.2: First set of simulations: Estimation of the second delayed state $x_2(t-\tau)$

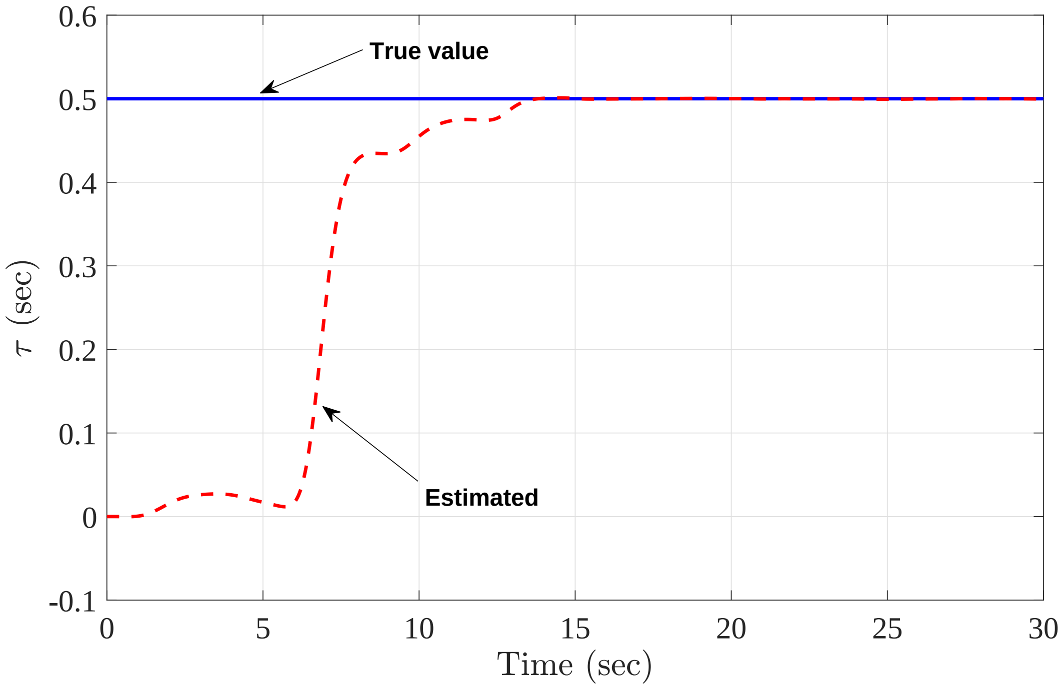

Figure 3.3: First set of simulations: Estimation of the unknown time-delay value τ

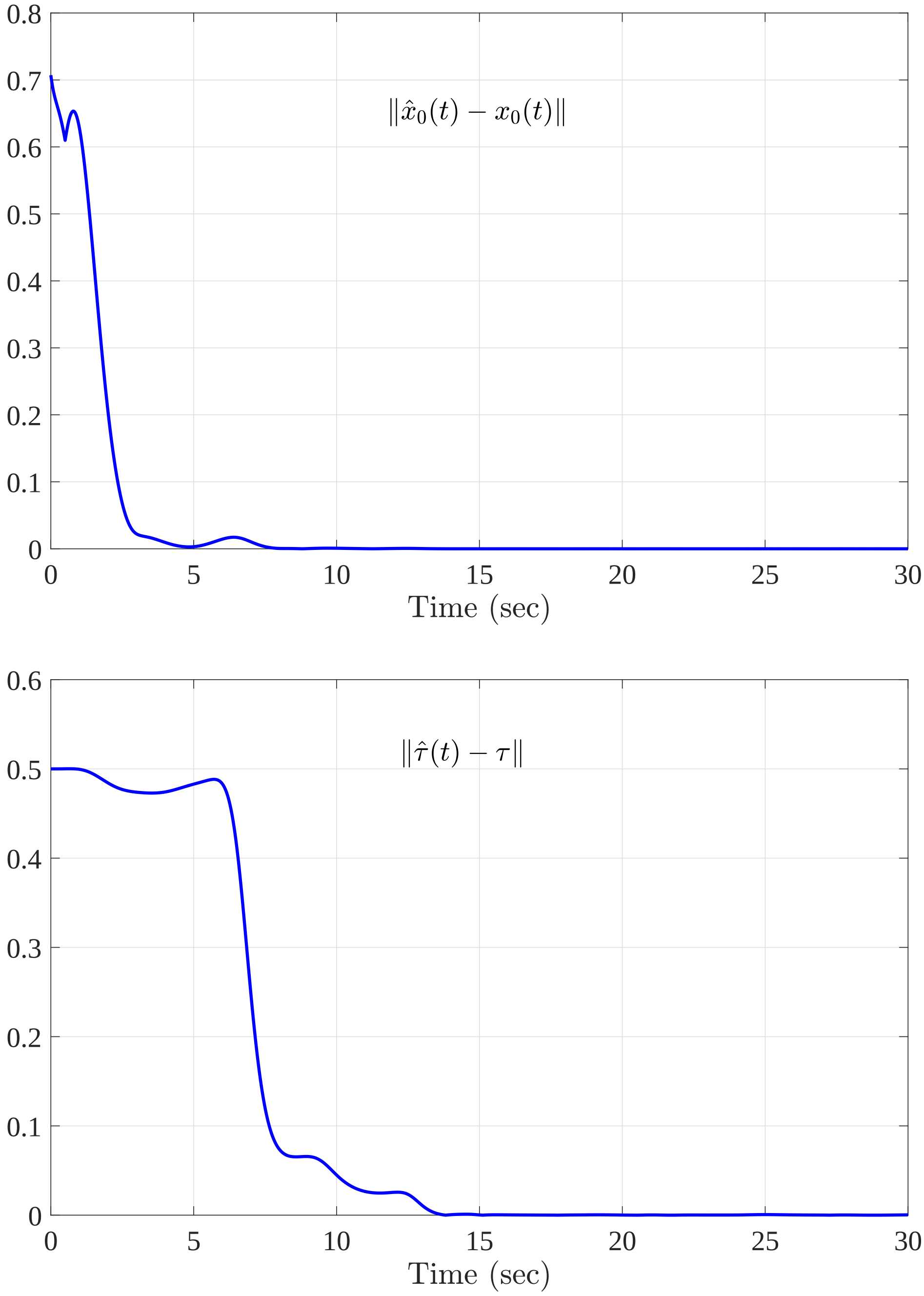

Figure 3.4: First set of simulations: Observation error norm of states and unknown time-delay in the head observer

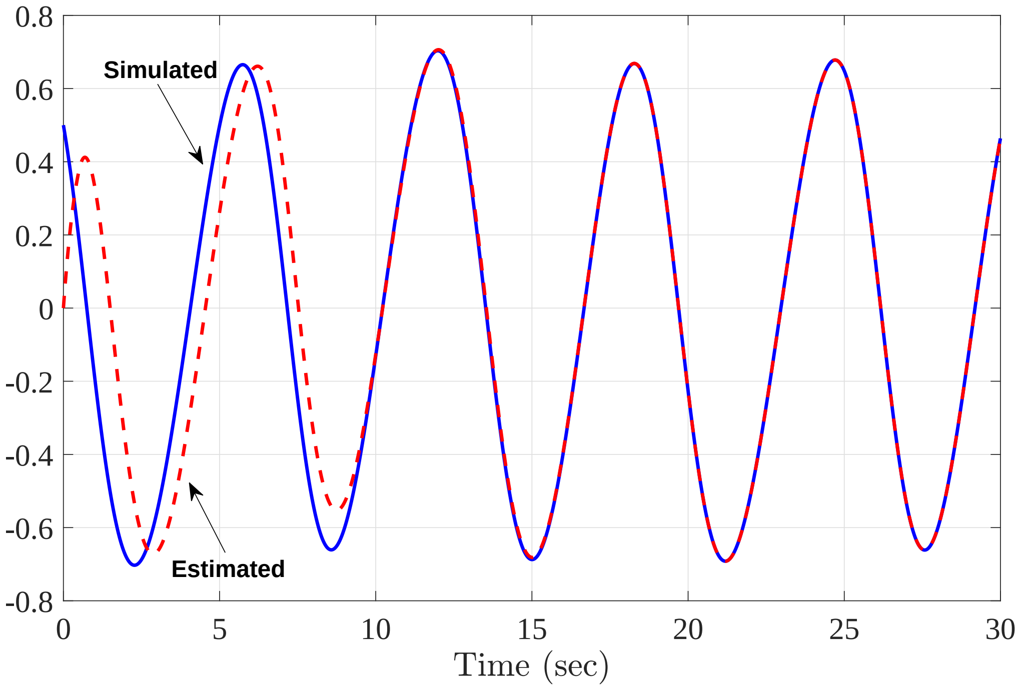

Figure 3.5: First set of simulations: Estimation of the first actual state $x_1(t)$

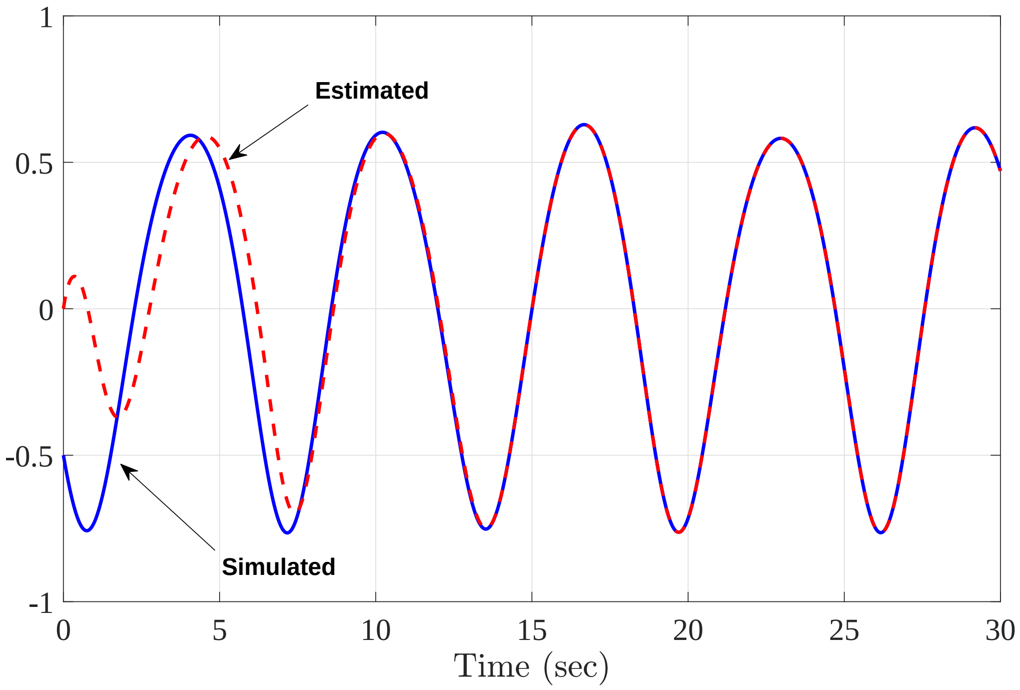

Figure 3.6: First set of simulations: Estimation of the second actual state $x_2(t)$

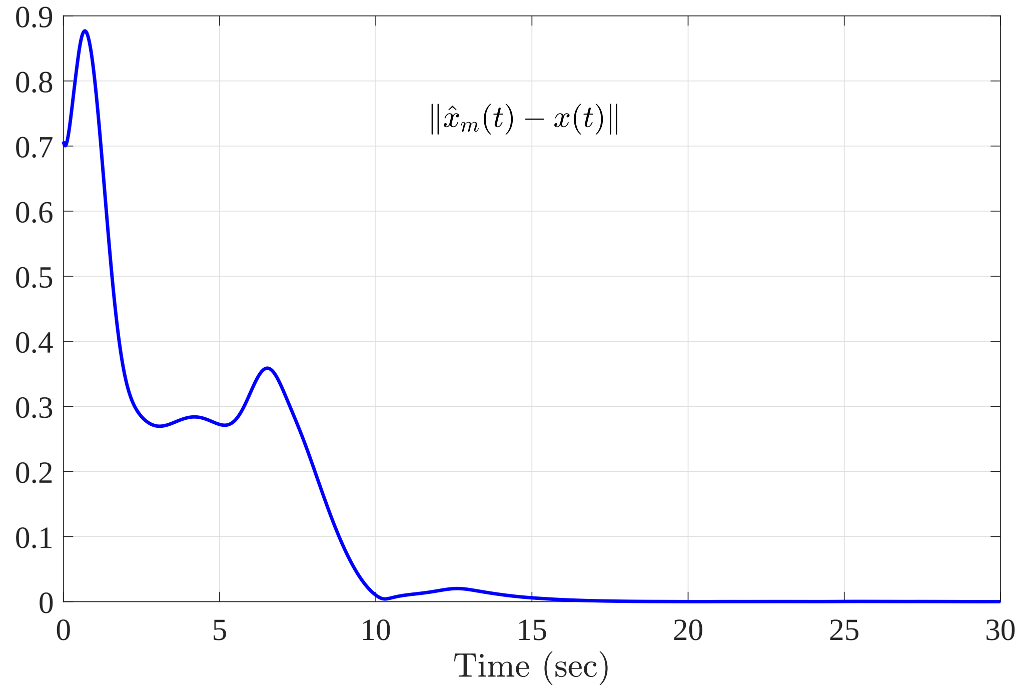

Figure 3.7: First set of simulations: Observation error norm of the actual state vector

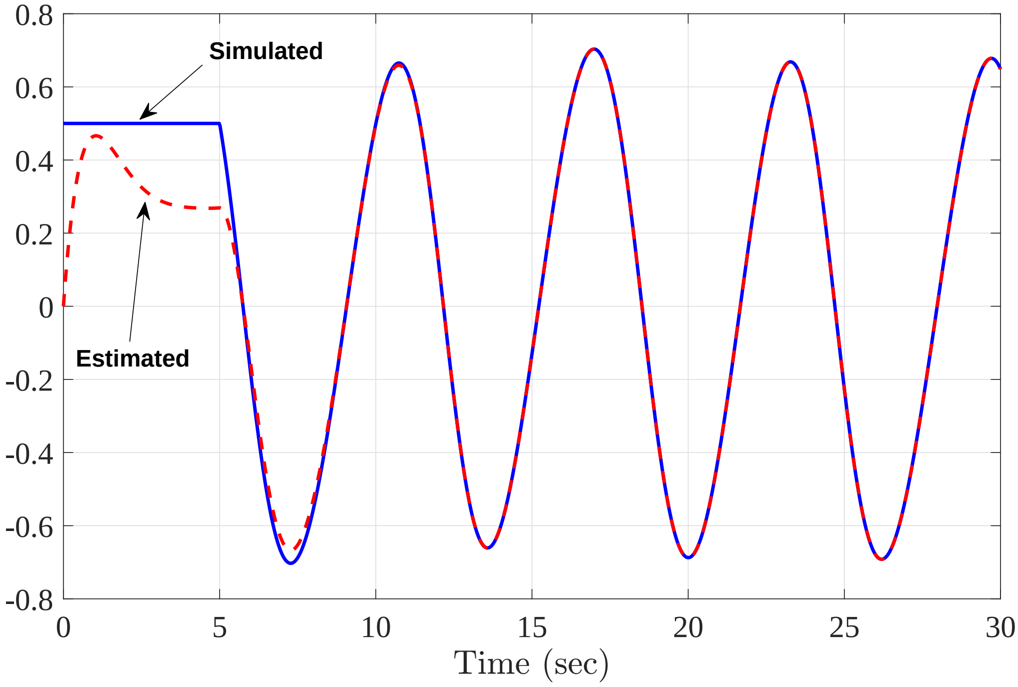

Figure 3.8: Second set of simulations: Estimation of the first delayed state $x_1(t - \tau)$

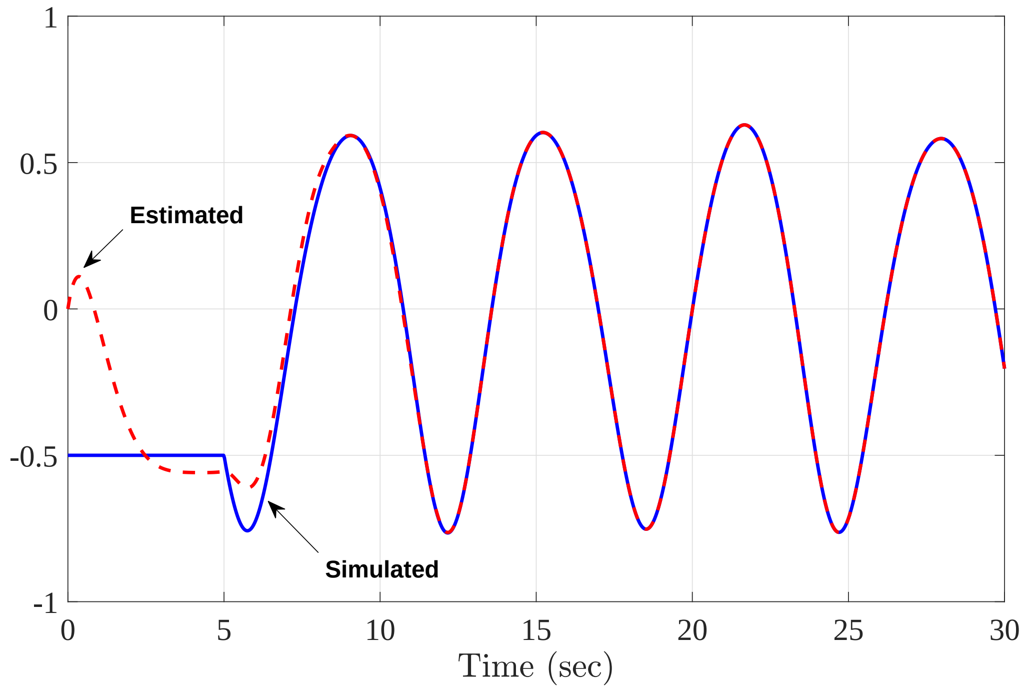

Figure 3.9: Second set of simulations: Estimation of the second delayed state $x_2(t-\tau)$

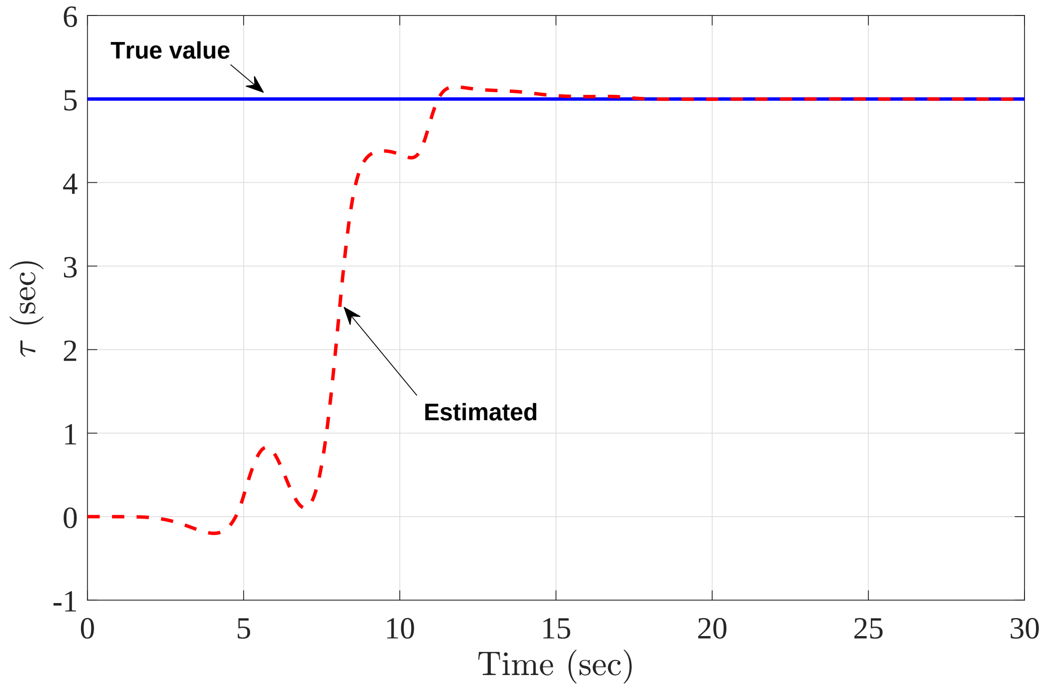

Figure 3.10: Second set of simulations: Estimation of the unknown time-delay value τ

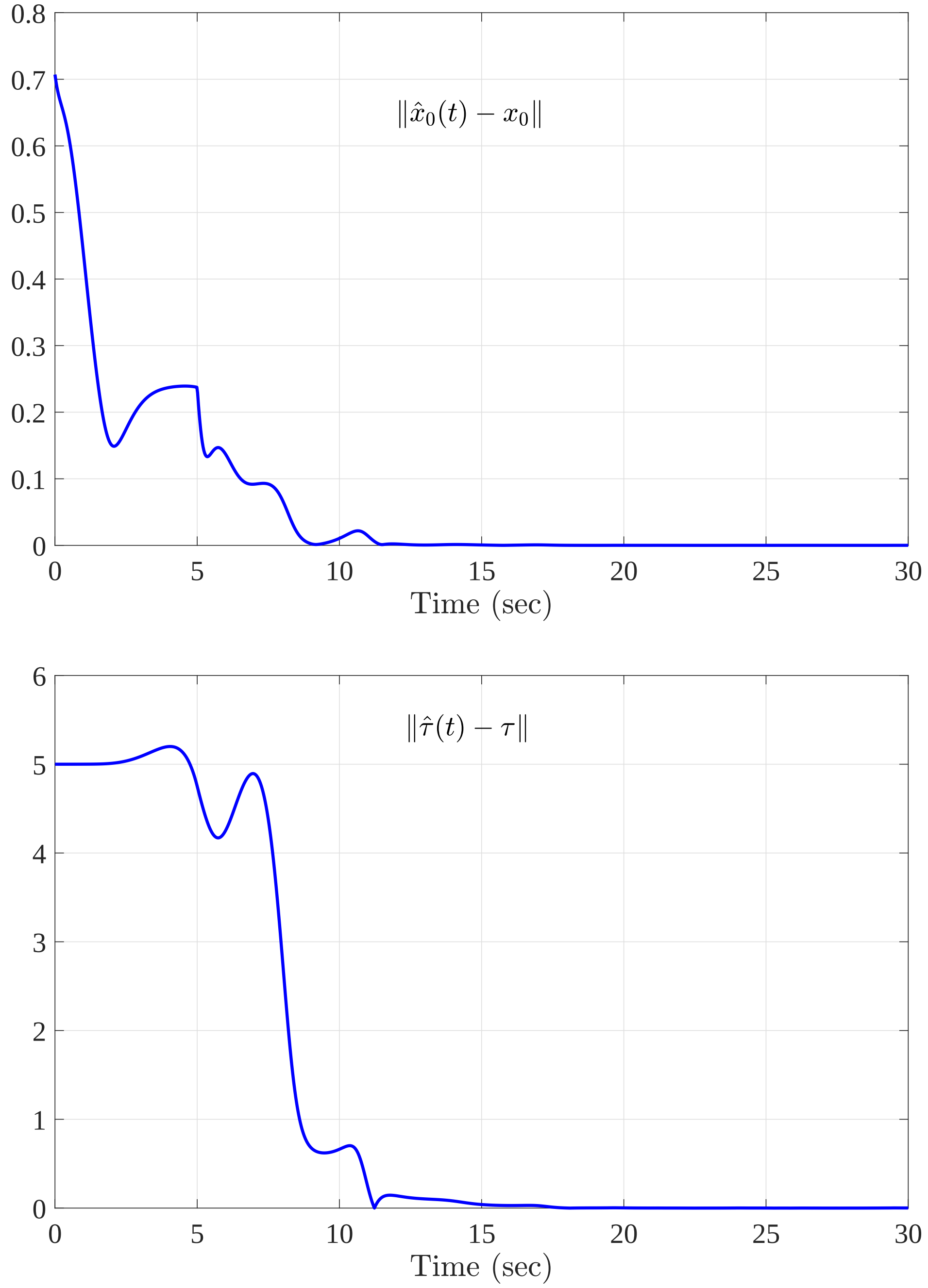

Figure 3.11: Second set of simulations: Observation error norm of states and unknown time-delay in the head observer

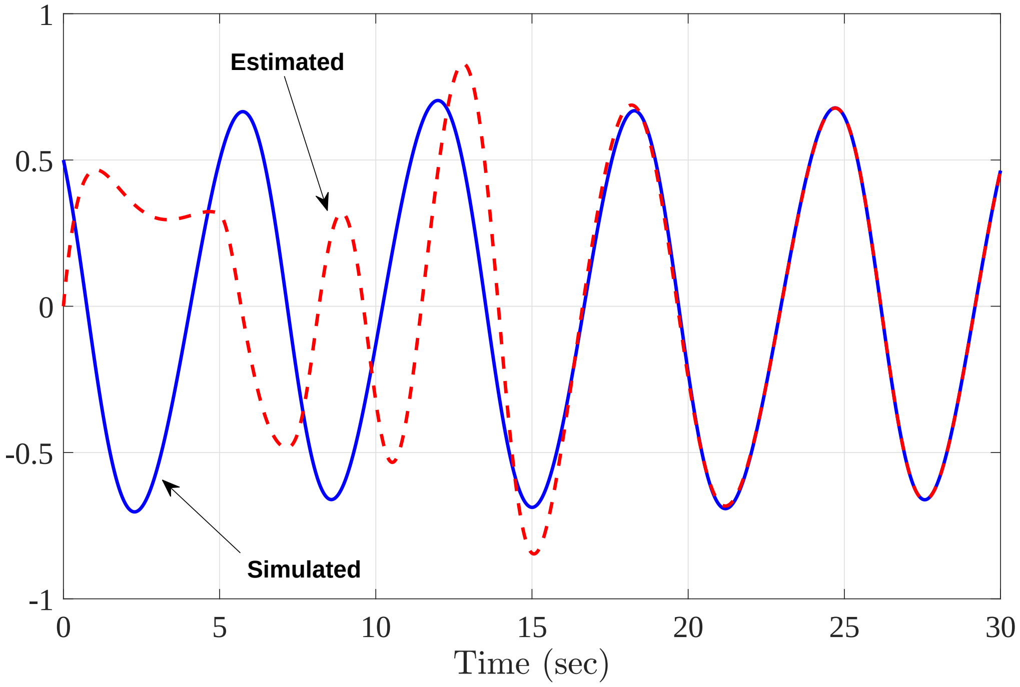

Figure 3.12: Second set of simulations: Estimation of the first actual state $x_1(t)$

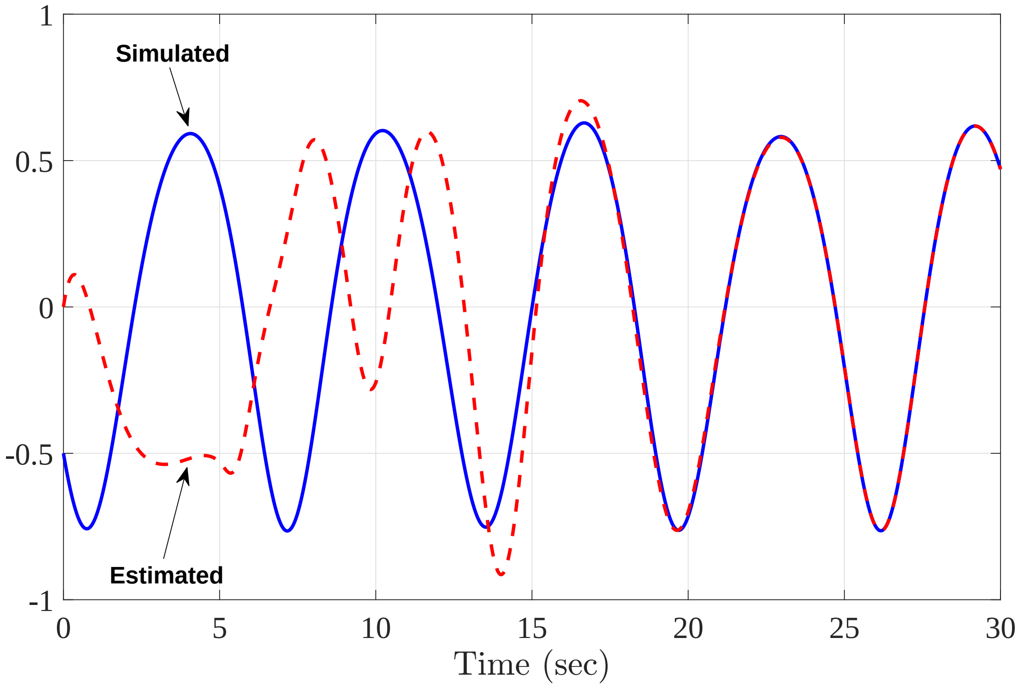

Figure 3.13: Second set of simulations: Estimation of the second actual state $x_2(t)$

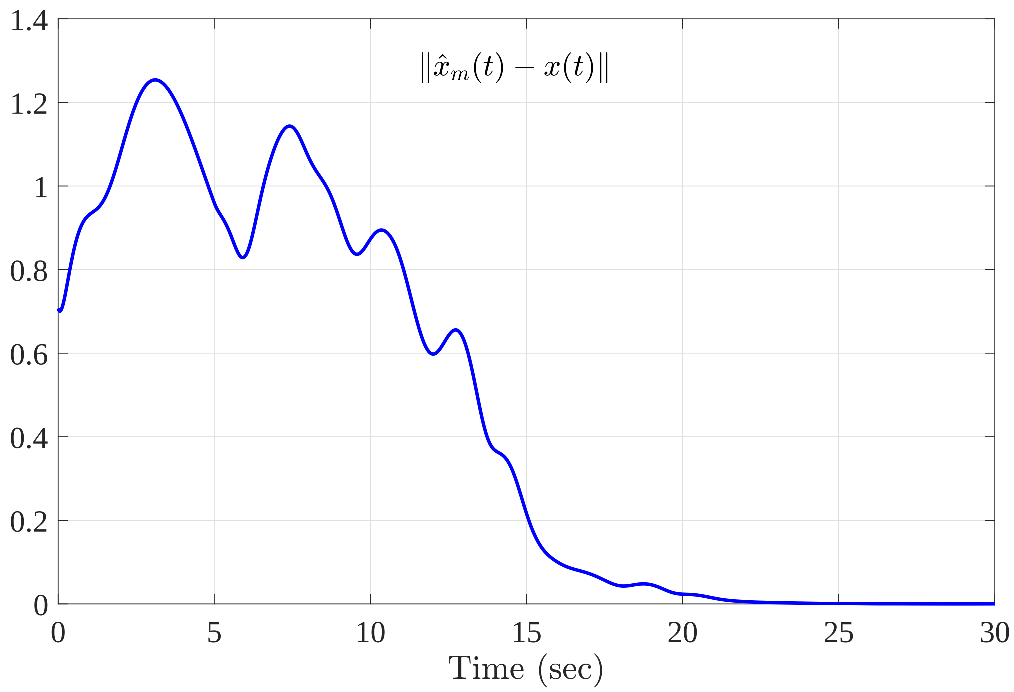

Figure 3.14: Second set of simulations: Observation error norm of the actual state vector

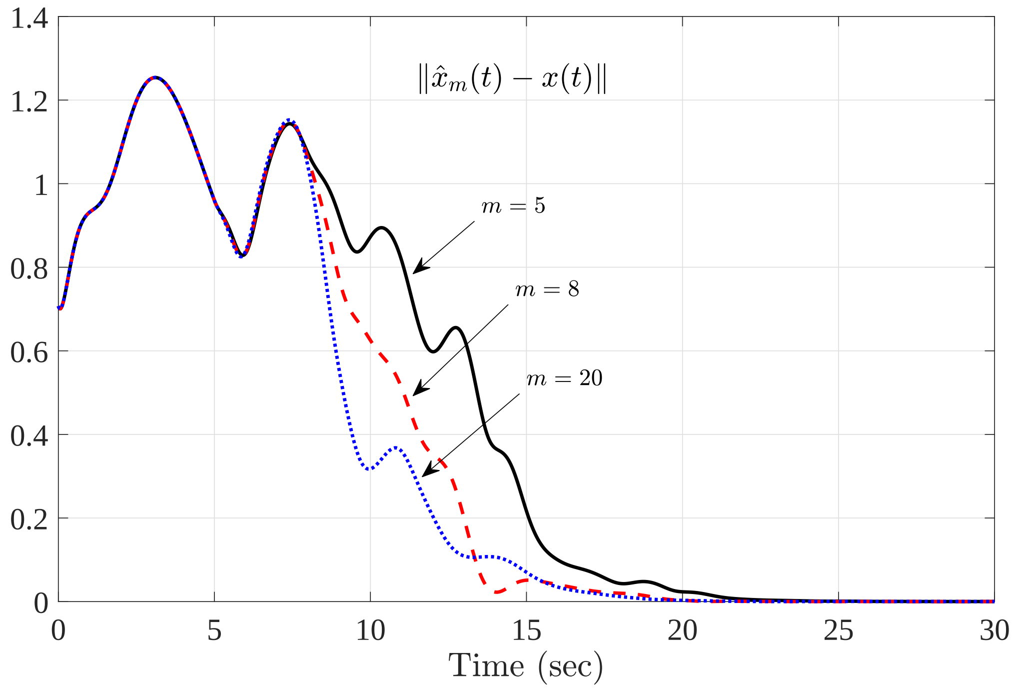

Figure 3.15: Third simulation: Comparison of the observation error norm with different number of predictors

3.5.2 Robotic arm

The following simulation demonstrates the operation of the observer applied to a real physical system.

One considers the following robotic arm system proposed in (Farza et al., 2021; Wang et al., 2017):

$$\mathcal{SYS}: \left\{ \begin{array}{rl} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = & \begin{bmatrix} x_2(t) \\ \frac{\bar{K}}{J_2 N} x_3(t) - \frac{F_2}{J_2} x_2(t) - \frac{\bar{K}}{J_2} x_1(t) - \frac{\bar{m}gd}{J_2} \cos(x_1(t)) \\ x_4(t) \\ \frac{1}{J_1} u - \frac{\bar{K}}{J_1 N} x_1(t) - \frac{\bar{K}}{J_2 N} x_3(t) - \frac{F_1}{J_1} x_4(t) \end{bmatrix} \\ y_\tau(t) = & Cx(t-\tau) = x_1(t-\tau) \end{array} \right. \tag{3.126}$$

where $x(t) = [x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)]^T \in \mathbb{R}^4$ denotes the actual state vector; $y_\tau(t) \in \mathbb{R}^p$ with $p = 1$ denotes the available output; the system input is denoted by $u(t) \in \mathbb{R}$. The available output presents an unknown time-delay $\tau > 0$.

The parameter values of system (3.126) are shown in Table 3.1.

Parameter	**Value**	**Unit**
J_1	0.15	kgm^2
J_2	0.2	kgm^2
$\bar{K}$	0.4	N/m
N	2	-
$\bar{m}$	0.8	kg
g	9.81	$\frac{\mathrm{m}}{\mathrm{s}^2}$
d	0.6	-
F_1	0.1	-
F_2	0.15	-
N	1400	-

Table 3.1: Physical parameters of the robotic arm

Considering the above, it is possible to rewrite system (3.126) as follows:

$$\mathcal{SYS}: \left\{ \begin{array}{rl} \dot{x}(t) = & Ax(t) + \varphi(u(t), x(t)) \\ y_\tau(t) = & Cx(t-\tau) = x_1(t-\tau) \end{array} \right. \tag{3.127}$$

with:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\bar{K}}{J_2 N} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \varphi(u(t), x(t)) = \begin{pmatrix} 0 \\ -\frac{F_2}{J_2}x_2(t) - \frac{\bar{K}}{J_2}x_1(t) - \frac{\bar{m}gd}{J_2}\cos(x_1(t)) \\ 0 \\ \frac{1}{J_1}u(t) - \frac{\bar{K}}{J_1 N}x_1(t) - \frac{\bar{K}}{J_2 N}x_3(t) - \frac{F_1}{J_1}x_4(t) \end{pmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \tag{3.128}$$

with $\frac{\bar{K}}{J_2 N} = 1$ and the system input $u(t) = \sin(t)$.

Now, the system (3.127) is in the form (3.1) and presents an unknown delay at the output. In order to estimate the actual state vector and the unknown time-delay, one designs a cascade observer governed by the equations (3.36).

Before proceedings, it is necessary to obtain some terms that are involved in the head of the observer ($j = 0$). According to Assumption 3.3.1 and equation (3.11), one obtains $\nu = 4$. Since in the vector of nonlinear functions $\varphi(u(t), x(t))$ of system (3.127), the input exists only in the last state, i.e. $k = 4$.

Now, considering $x_0(t) = x(t - \tau)$, the nonlinear vector function can be rewritten as follows:

$$\varphi(\hat{x}_0(t), u(t - \hat{\tau})) = \begin{pmatrix} 0 \\ -\frac{F_2}{J_2}x_{2,0}(t) - \frac{\bar{K}}{J_2}x_{1,0}(t) - \frac{\bar{m}gd}{J_2}\cos(x_{1,0}(t)) \\ 0 \\ \frac{1}{J_1}u(t - \hat{\tau}) - \frac{\bar{K}}{J_1 N}x_{1,0}(t) - \frac{\bar{K}}{J_2 N}x_{3,0}(t) - \frac{F_1}{J_1}x_{4,0}(t) \end{pmatrix} \tag{3.129}$$

Bearing in mind the above, one obtains $\Phi(u(t - \hat{\tau}), \hat{z})$ as follows:

$$\begin{aligned} \Phi(u(t - \hat{\tau}), \hat{x}_0(t)) &= -\frac{\partial}{\partial u}\varphi^s(u(t - \hat{\tau}(t)), \underline{\hat{x}_{0,\nu}})\dot{u}(t - \hat{\tau}(t)), \\ &= -\frac{1}{J_1}\cos(t - \hat{\tau}(t)) \end{aligned} \tag{3.130}$$

and according to equation (3.11) the column vector B_ν is

$$B_\nu(\nu) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{3.131}$$

Two sets of simulations are presented below to demonstrate different scenarios that the observer can deal with. All simulations will have the following initial conditions and parameters. Note that all simulations were obtained without the saturation of nonlinearities and state.

The matrix S is the unique solution to the equation (3.14) and Δ_θ are as follows:

$$S = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 2 & -3 & 4 \\ 1 & -3 & 6 & -10 \\ -1 & 4 & -10 & 20 \end{bmatrix}, \ \Delta_\theta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\theta} & 0 & 0 \\ 0 & 0 & \frac{1}{\theta^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\theta^3} \end{bmatrix}, \tag{3.132}$$

and the tuning parameter is set to $\theta = 2$. The initial conditions for the original system are:

$$x(0) = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} \tag{3.133}$$

for the head observer ($j = 0$):

$$\hat{x}(s) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \forall s \in [-\hat{\tau}(0), 0], \Upsilon(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\hat{\tau}(0) = 0, \ p(0) = 0.01, \tag{3.134}$$

and for the predictors ($j = 1, \ldots, m$):

$$\hat{x}_j(s) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \forall s \in [-\hat{\tau}(0), 0], \ \bar{A} = A - \lambda I_{2\times 2} \tag{3.135}$$

and the number of predictors is equal to $m = 10$.

The first set of simulation is considering $\lambda = 1$. Figures 3.16-3.26 show the results of the first simulation.The estimation of the delayed state vector by the observer at the head of the cascade is shown in Figures 3.16-3.19. Similarly, this observer achieves the estimation of the unknown delay value as shown in Figure 3.20. It is clear that the states and the delay value converge to the original ones, as shown in Figure 3.21, where the norm of the errors is shown. This shows that the observation error of the states and the parameter converges exponentially to zero. The actual state vector estimated by the last predictor $x_j(t), \ j = m = 10$ in the cascade is shown in Figures 3.22-3.25. The norm of the state observation error is shown in Figure 3.26. Thus, it is clear that the designed observer is

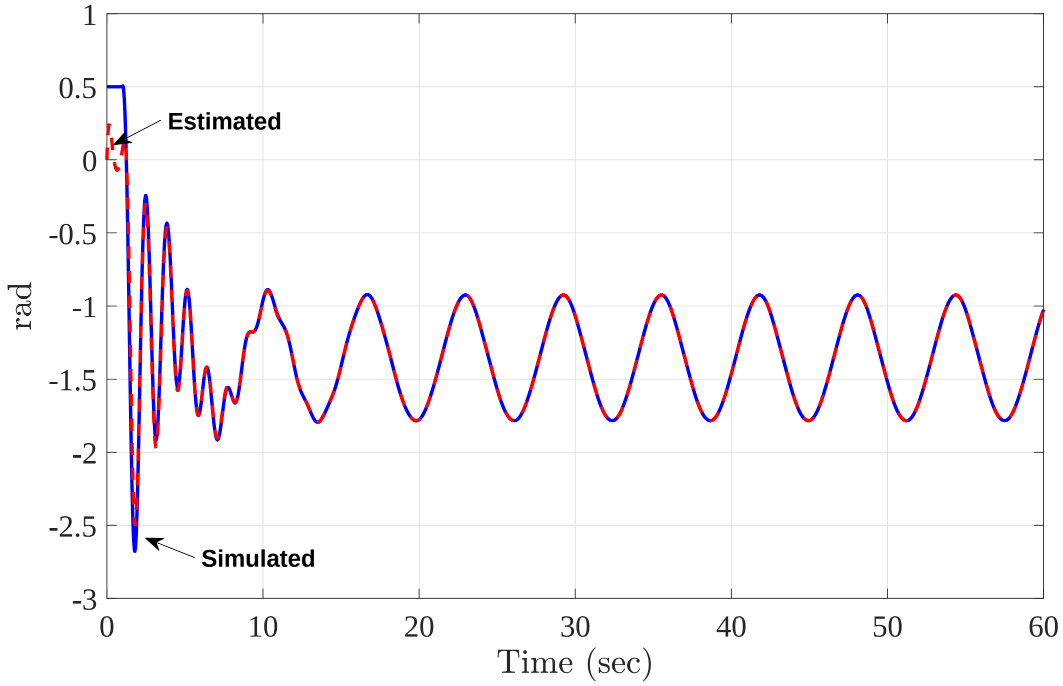

Figure 3.16: First set of simulations: Estimation of the first delayed position $x_1(t-\tau)$

able to estimate the current state vector even in the presence of an unknown delay in the output of the original system.

The second set of simulations is to show the performance of the cascade predictors by changing the values of the tuning parameter λ. For this purpose, three values $\lambda = 1, 3, 50$ are considered. The initial conditions are kept the same for all simulations, and for better demonstration, only the observation error of the last cascade predictor is shown. Figure 3.27 shows that for larger values of λ a smoother convergence is obtained with fewer oscillations in the transient compared to smaller values.

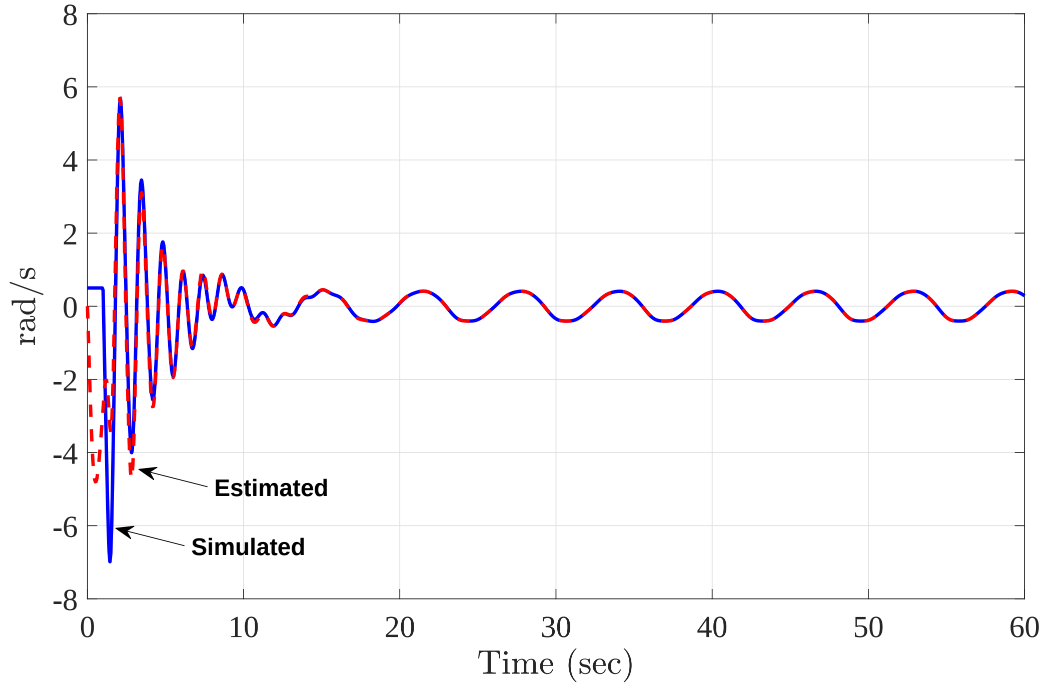

Figure 3.17: First set of simulations: Estimation of the first delayed velocity $x_2(t-\tau)$

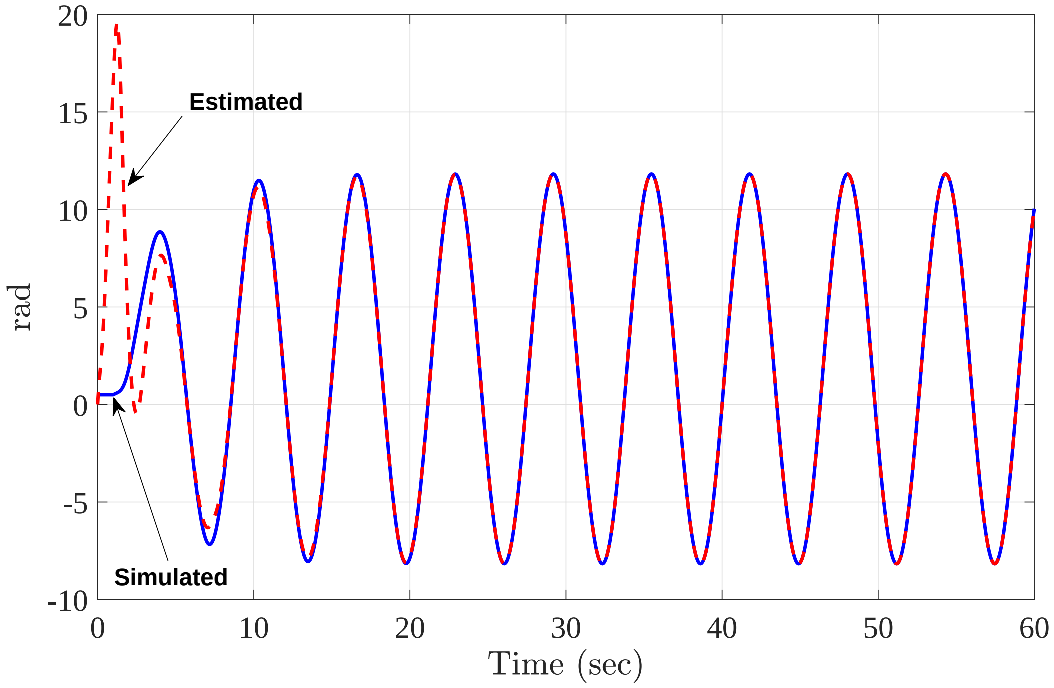

Figure 3.18: First set of simulations: Estimation of the second delayed position $x_3(t-\tau)$

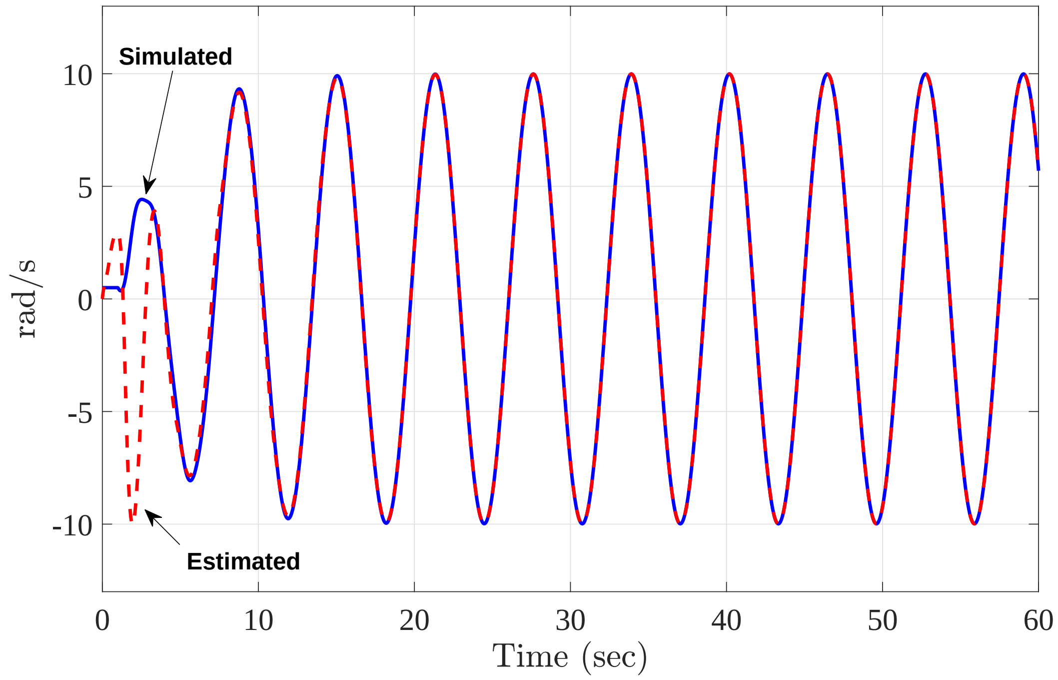

Figure 3.19: First set of simulations: Estimation of the second delayed velocity $x_4(t - \tau)$

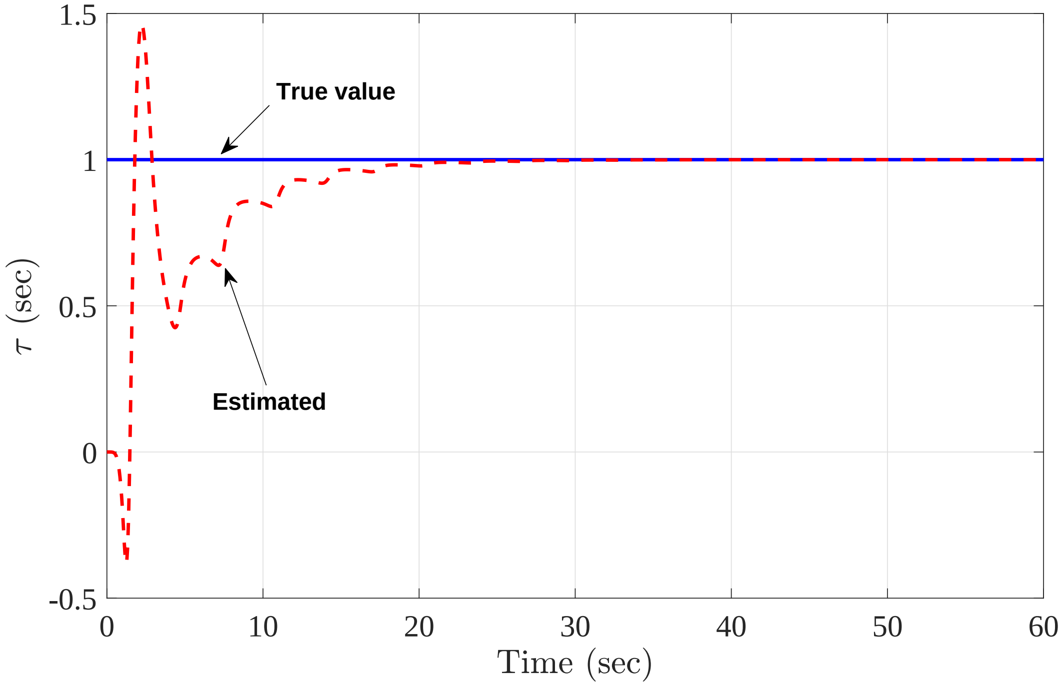

Figure 3.20: First set of simulations: Estimation of the unknown time-delay value τ

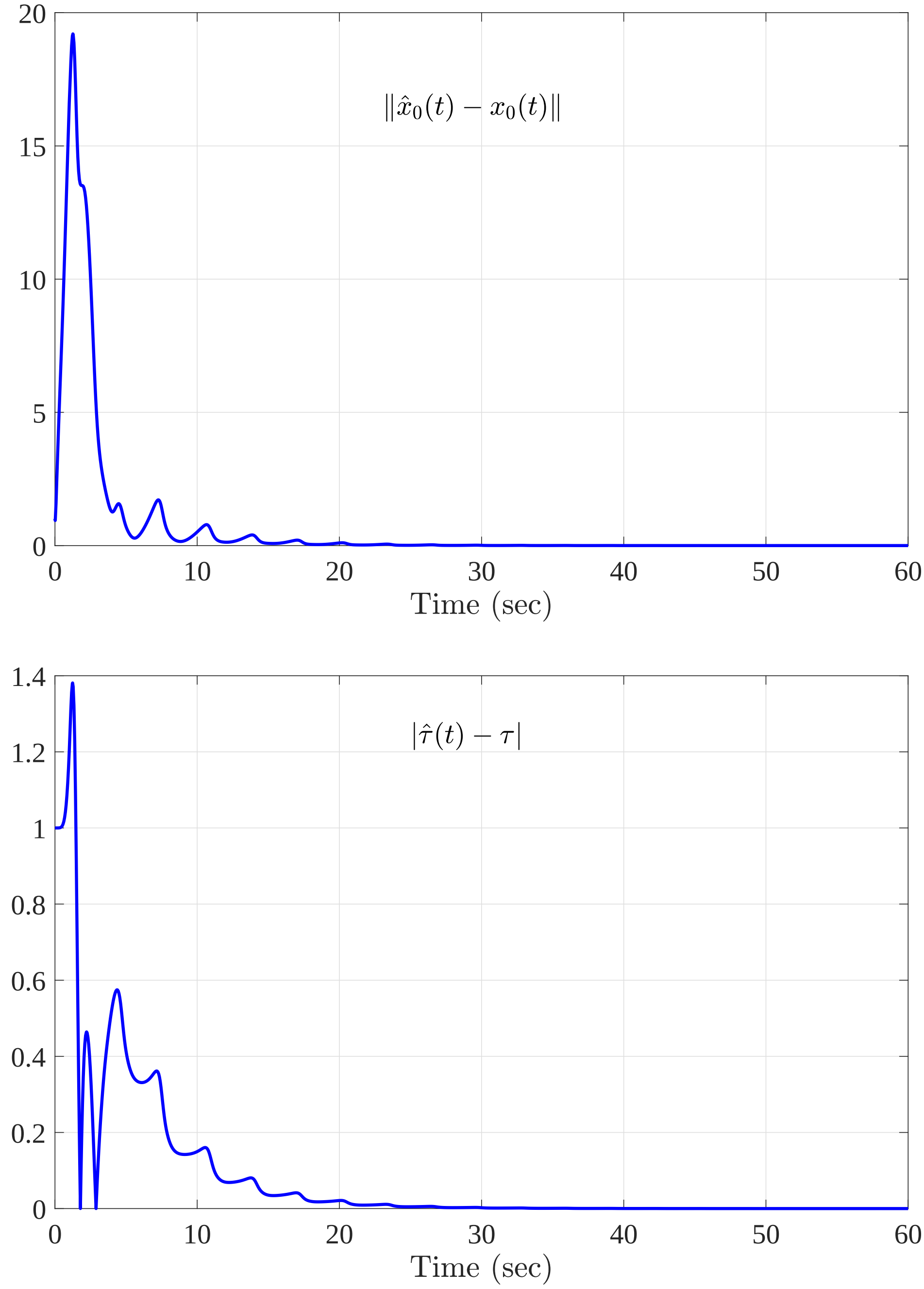

Figure 3.21: First set of simulations: Observation error norm of states and unknown time-delay in the head observer

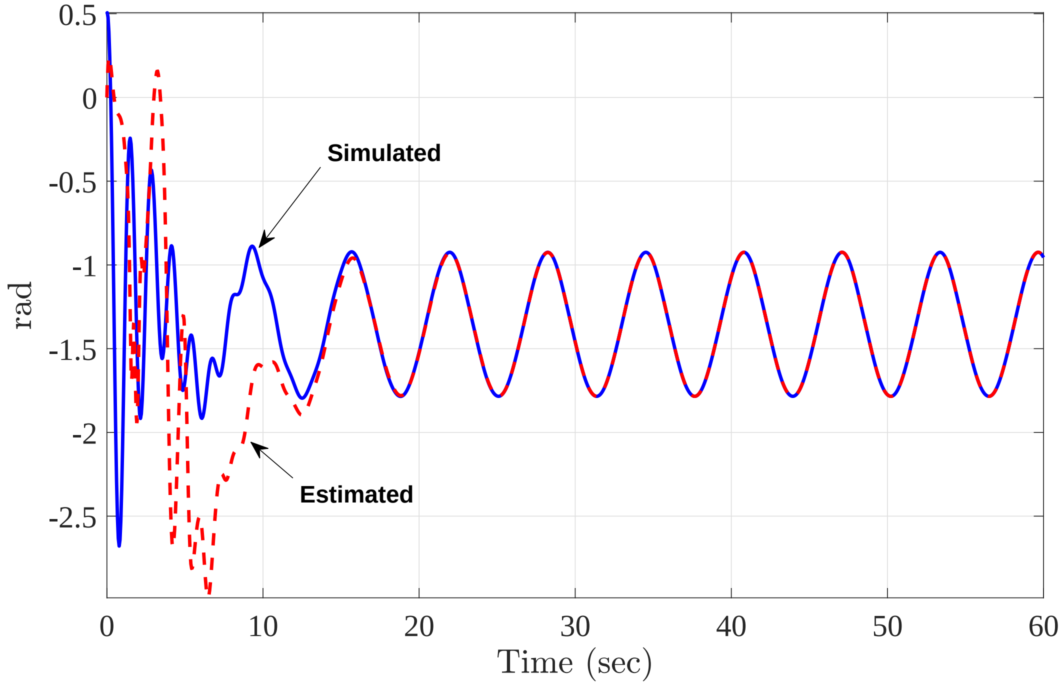

Figure 3.22: First set of simulations: Estimation of the actual first position $x_1(t)$

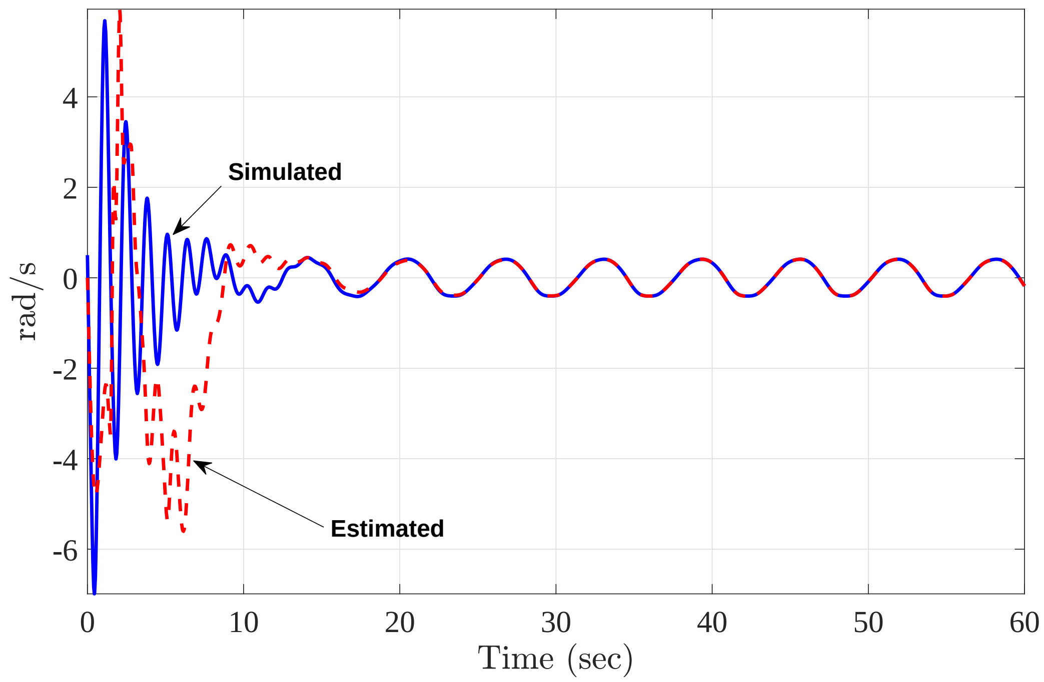

Figure 3.23: First set of simulations: Estimation of the actual first velocity $x_2(t)$

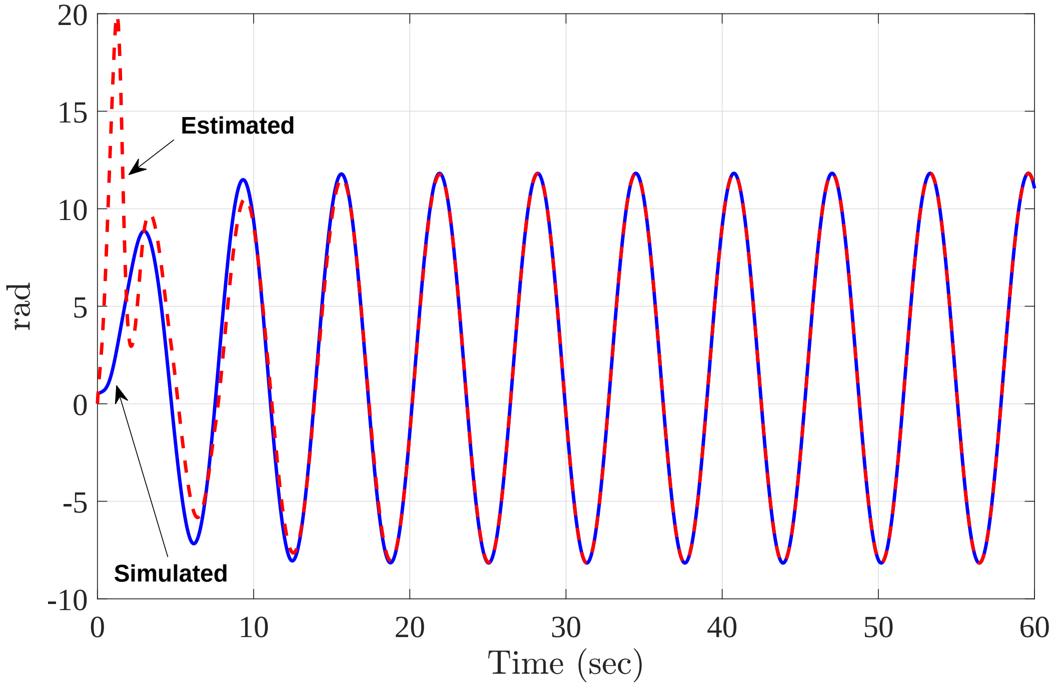

Figure 3.24: First set of simulations: Estimation of the actual second position $x_3(t)$

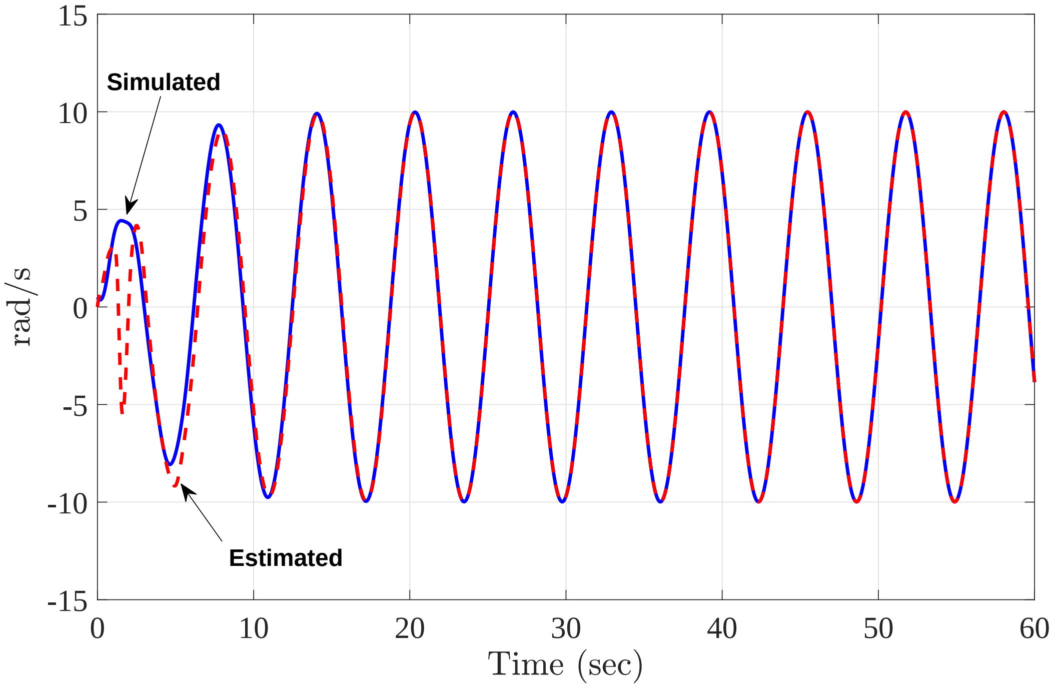

Figure 3.25: First set of simulations: Estimation of the actual second velocity $x_4(t)$

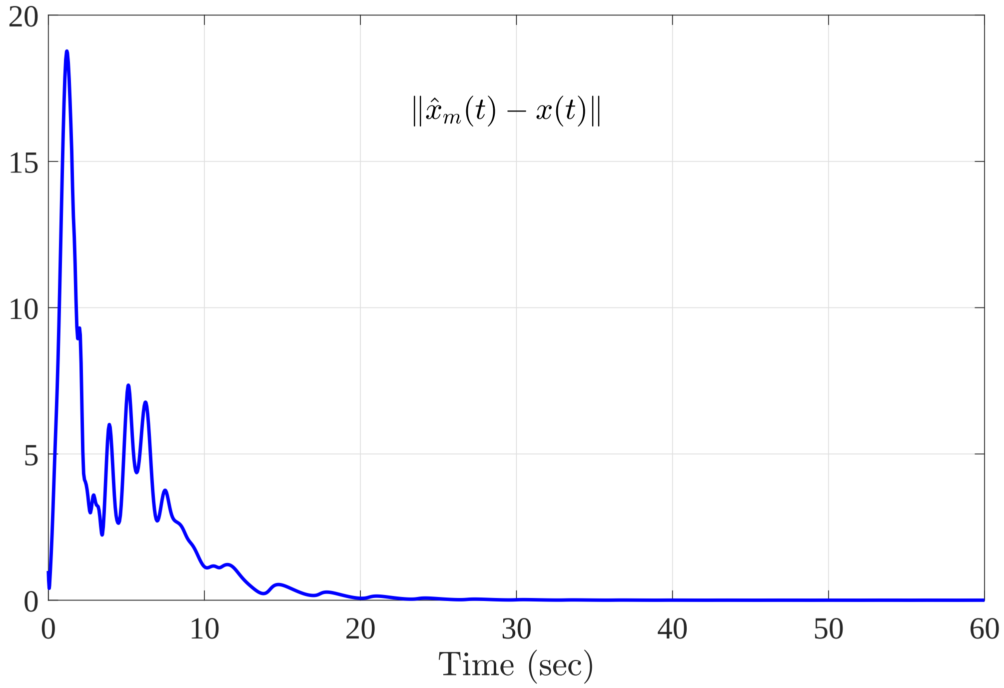

Figure 3.26: First set of simulations: Observation error norm of the actual state vector

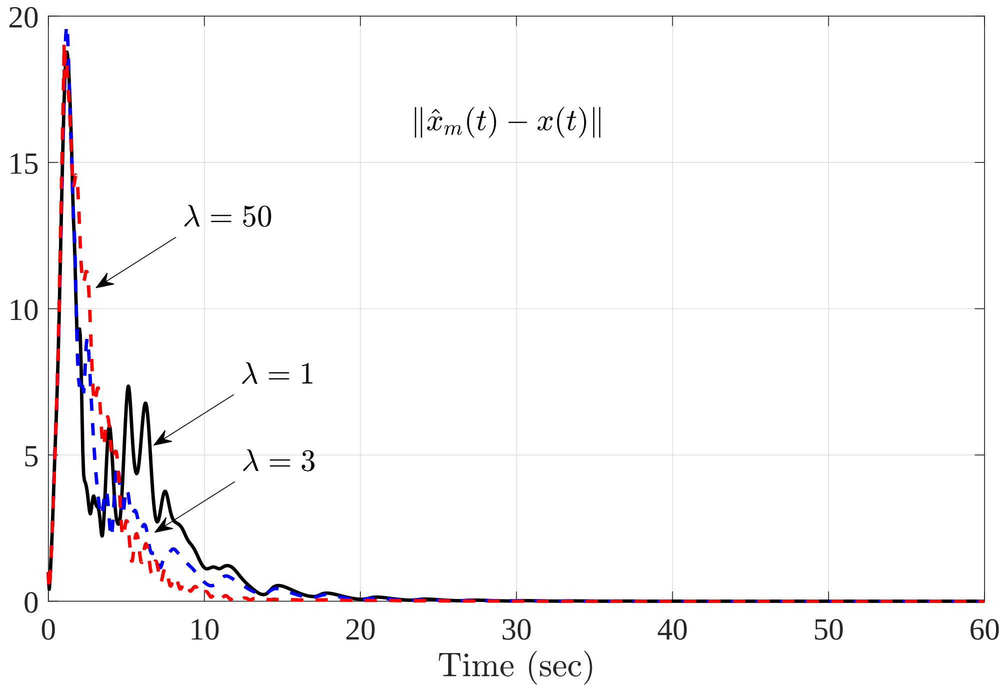

Figure 3.27: Second set of simulation: Comparison of the observation error norm with different number of λ

3.6 Conclusion

This chapter has presented the design of an observer for a class of nonlinear systems in the presence of an unknown time delay in the output. This problem was approached by considering the use of a cascaded observer, which is capable of jointly estimating the actual state vector and the value of the unknown delay present at the output. First, the redesign of the observer in the observer head is presented and an adaptive structure is proposed, achieving the estimation of the system delay vector and the value of the unknown delay. Then, based on the estimation of the unknown delay, the predictors are able to estimate the original system and predict it within a fraction of the delay. The methodology for the design of the head observer and the selection of the number of predictors is clearly presented. It is shown that the estimation error of states and in the unknown time delay will converge to zero. The effectiveness of the observer is demonstrated through an academic example and a robotic arm.

Chapter 4

Observer synthesis for a class of nonlinear with state and output delays

4.1 Introduction

In this chapter, the design of an observer for uncertain uniformly observable systems in the presence of long output and state delays will be discussed. The observer to be designed presents a cascade structure due to the long output delay. Two main cases are considered for the design of the observer: when the state vector related to the linear part is free of delay and when there are states with delay in it. In addition, the methodology to reconstruct the original states from the cascade of observers is presented, considering the different types of delay: commensurable and non commensurable.

There are works that choose to separate the linear part without delay and the part with delay, as proposed by (Binazadeh and Asadinia, 2020; Naami et al., 2022; Avilés et al., 2023). However, these approaches do not consider delays in the system output, which in practice is common due to the measurement and data transmission instruments. An approach that addresses output delays and states is presented in (Vafaei and Yazdanpanah, 2016; Talebi et al., 2020; Di Ferdinando et al., 2023). However, this approach is limited to the linear dynamics being delay-free and the delayed states being present in the nonlinear part.

The performance of the proposed cascaded observer is highlighted by simulations on an academic example and a simple pendulum.

4.2 Problem formulation

Let us consider the following class of uniformly nonlinear systems with time-delay in state and output:

$$\mathcal{SYS}: \begin{cases} \dot{x}(t) &= Ax(t) + \varphi(u(t), x(t), x_{\tau_s}(t)) + B\varepsilon(t) \\ y_a(t) &= Cx(t-\tau_1) = x_1(t-\tau_1), \\ x(t) &= \psi_0(t), \forall t \in [-\tau_1 - \tau_s, 0], \end{cases} \tag{4.1}$$

with

$$A = \begin{bmatrix} 0_{n-1,1} & I_{n-1} \\ 0 & 0_{1,n-1} \end{bmatrix} \in \mathbb{R}^{n\times n}, B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{n\times 1}, C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{1\times n}, \tag{4.2}$$

where $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_n(t) \end{bmatrix}^T \in \mathbb{R}^n$ is the actual state vector of the system; $x_{\tau_s}(t) = x(t-\tau_s)$ is its delayed version and τ_s is the delay present in the state; $u(t) \in \mathbb{R}$ is the system input; $y_a(t) = y(t-\tau_1) \in \mathbb{R}$ denotes the available delayed output where τ_1 represents the time-delay present in the output, $y(t) \in \mathbb{R}$ is the actual system output that is not available; $\varepsilon : [-\tau_1, +\infty[\mapsto \mathbb{R}$ denotes the unknown function that represents the uncertainties of the system. It shall be treated as an unknown function which explicitly depends on the time t for $t \geq -\tau_1$; $\psi_0(t)$ is a continuous function $[-\tau_1 - \tau_s, 0]$ and $\varphi(u(t), x(t), x_{\tau_s}(t))$ is a nonlinear vector function with a triangular structure with respect to the state $x(t)$ and its delayed version $x_{\tau_s}(t)$, i.e.

$$\varphi_k(u(t), x(t), x_{\tau_s}(t)) = \varphi_k(u(t), x_1(t), \dots, x_k(t), x_1(t-\tau_s), \dots, x_k(t-\tau_s)), k = 1, \dots, n. \tag{4.3}$$

Before proceedings one defines the following assumptions for high gain observer design (Hernández-González et al., 2016; Farza et al., 2017a, 2014b):

A1. The input system $u(t)$ and state $x(t)$ are bounded, i.e. there exist compact sets $U \subset \mathbb{R}$ and $X \subset \mathbb{R}^n$ such that $u(t) \in U$ and $x(t) \in X$ $\forall t \geq 0$, allowing thereby to define an upper bound on the system state variables, namely there exists a positive scalar ρ_M such that one has

$$\forall t \geq 0, |x_i(t)| \leq \rho_M \text{ for } i = 1, \dots n \tag{4.4}$$

A2. The nonlinearity φ is Lipschitz with respect to the system state $x(t)$ and its delayed version x_{τ_s}, uniformly in the system input $u(t)$, i.e. there exists a positive scalar L_φ such that $\forall u \in U; \forall (x, \bar{x}) \in X \times X; \forall (x_{\tau_s}, \bar{x}_{\tau_s}) \in X \times X$, one has

$$\|\varphi(u(t), x(t), x_{\tau_s}(t)) - \varphi(u(t), \bar{x}(t), \bar{x}_{\tau_s}(t))\| \leq L_\varphi \|x(t) - \bar{x}(t)\| + L_\varphi \|x_{\tau_s}(t) - \bar{x}_{\tau_s}(t)\|. \tag{4.5}$$

A3. The unknown function $\varepsilon(t)$ is essentially bounded, i.e.

$$\exists \delta_\varepsilon > 0; \sup_{t \geq -\tau_1} ess\|\varepsilon(t)\| \leq \delta_\varepsilon. \tag{4.6}$$

A4. The output time-delay τ_1 is arbitrarily long, constant, and known.

The estimation strategy is inspired by the articles Hernández-González et al. (2016); Farza et al. (2018a); Germani et al. (2002), where a cascade observer is presented for the estimation of the actual state vector under arbitrary long time-delay. The structure of this observer consists of $m+1$ subsystems. The first subsystem estimates the delayed state vector and the remaining m subsystems predict the system over a prediction horizon equal to $\frac{\tau_1}{m}$, whereupon the last subsystem reconstructs the current state vector.

One shall first detail the design of the High Gain Observer (HGO) which constitutes the head of the cascade. Before proceeds one saturates the nonlinealities of the system. Indeed, let us recall the usual saturation function defined as follows:

$$\forall z <\in \mathbb{R}, sat_r(z) = r \cdot sat\left(\frac{z}{r}\right) = \begin{cases} z & \text{if } |z| \leq r \\ r \cdot sign(z) & \text{if } |z| > r \end{cases} \tag{4.7}$$

where $r > 0$ is a positive real and $sign(\cdot)$ is the usual signum function. Indeed, one can saturate the nonlinear functions φ_k for $k = 1, \ldots, n$ as follows:

$$\begin{aligned} &\varphi_k^s(u(t), x(t), \ldots, x_k(t), x_1(t-\tau_s), \ldots, x_k(t-\tau_s)) \triangleq \\ &\varphi_k^s(u(t), sat_R(x(t)), \ldots, sat_R(x_k(t)), sat_R(x_1(t-\tau_s)), \ldots, sat_R(x_k(t-\tau_s))), \end{aligned} \tag{4.8}$$

with $R > \rho_M$ and where ρ_M is the positive real defined in assumption **A1**. Since the Lipschitz constant of the saturation function is equal to one, the Lipschitz condition (4.5) implies that

$$\begin{aligned} &\forall u \in U; \forall (x, \bar{x}) \in X \times X; \forall (x_{\tau_s}, \bar{x}_{\tau_s}) \in X \times X \\ &\|\varphi(u(t), x(t), x_{\tau_s}(t)) - \varphi(u(t), \bar{x}(t), \bar{x}_{\tau_s}(t))\| \leq L_\varphi(\|x(t) - \bar{x}(t)\| + \|x_{\tau_s}(t) - \bar{x}_{\tau_s}(t)\|). \end{aligned} \tag{4.9}$$

Recall that the saturated nonlinearities coincide with the non saturated ones on the domain in X in which lies the state, one can use the following system to design the high gain observer:

$$\mathcal{SYS}: \begin{cases} \dot{x}(t) &= Ax(t) + \varphi^s(u(t), x(t), x_{\tau_s}(t)) + B\varepsilon(t) \\ y_a(t) &= x_1(t - \tau_1) \end{cases} \tag{4.10}$$

4.3 Design of the high gain observer

Let us consider the following state vector $z(t) \triangleq x(t - \tau_1)$ as the delayed state of $x(t)$. One can estimate the state vector $z(t)$ using the following HGO:

$$\mathcal{OBS}: \begin{cases} \dot{\hat{z}}(t) &= A\hat{z}(t) + \varphi^s(u_{\tau_1}(t), \hat{z}(t), \hat{z}_{\tau_s}(t)) - \theta\Delta_\theta^{-1}K\left(C\hat{z}(t) - y_a(t)\right) \\ \hat{z}(s) &= \phi(s - \tau_1), \forall s \in [-\tau_s, 0], \end{cases} \tag{4.11}$$

with

$$\Delta_\theta = diag\left[1, \frac{1}{\theta}, \ldots, \frac{1}{\theta^{n-1}}\right], \tag{4.12}$$

where $\hat{z}(t) \in \mathbb{R}^n$ denotes the actual state estimate, $\hat{z}_{\tau_s}(t) \triangleq \hat{z}(t - \tau_s)$ denotes the delayed state estimate, $u_{\tau_1}(t) \triangleq u(t - \tau_1)$ denotes the delayed input, $\phi(s)$ is an arbitrary initial estimate of $x(s)$ for $s \in [-\tau_s - \tau_1, 0]$, $\theta > 0$ is a positive design parameter and K is chosen such that the matrix

$$\tilde{A} = A - KC, \tag{4.13}$$

is Hurwitz.

Before starting the theorem that puts forward the main property of observer (4.11), one recalls that since $\tilde{A}$ is Hurwitz, there exist a Symmetric Positive Define (SPD) matrix P and a positive real μ such that

$$P\tilde{A} + \tilde{A}P \leq -2\mu I_n. \tag{4.14}$$

Bearing in mind this, one can propose the following theorem.

Theorem 4.3.1 *Consider system (4.10) that satisfies the Assumptions* ***A1-A3*** *together with the observer (4.11). Then, one has:*

$$\begin{aligned} &\exists\theta_0 > 0; \forall\theta > \theta_0; \forall t \geq t_0, \\ &\|\tilde{z}(t)\| \leq \mu(\theta)e^{-\lambda(\theta)t} \max_{s\in[-\tau_s,0]} \|\tilde{z}(s)\| + M(\theta)\delta_\varepsilon \end{aligned} \tag{4.15}$$

with

$$\lambda(\theta) = \frac{ln(\theta)}{2\tau_s}, \mu(\theta) = \sigma_P\theta^{n-1}\left(1 + 2b\tau_s\frac{(\sqrt{\theta}-1)}{ln(\theta)}\right) \text{ and } M(\theta) = 2\sigma_P\frac{\tau_s}{ln(\theta)} \tag{4.16}$$

where $\tilde{z}(t) = \hat{z}(t) - z(t)$ is the observation error, σ_P is the conditioning number of the matrix P given by equation (4.14), τ_s is the time-delay involved in state of system (4.11), δ_ε is the essential bound of the unknown function $\varepsilon(t)$ and b is a positive constant independent of θ.

4.3.1 Proof of Theorem 4.3.1

Let us consider $z(t) = x(t-\tau_1)$ and $\tilde{z}(t) = \hat{z}(t) - z(t)$ as the observation error. From (4.10) and (4.11), one gets

$$\begin{aligned}
\dot{\tilde{z}}(t) &= \dot{\hat{z}}(t) - \dot{z}(t) \\
\dot{\tilde{z}}(t) &= A\hat{z}(t) + \varphi^s(u_{\tau_1}(t), \hat{z}(t), \hat{z}_{\tau_s}(t)) - \theta\Delta_\theta^{-1}K\left(C\hat{z}(t) - y_a(t)\right) \\
&\quad - \left[A\hat{z}(t) + \varphi^s(u_{\tau_1}(t), z(t), z_{\tau_s}(t))\right] \\
\dot{\tilde{z}}(t) &= A\tilde{z}(t) + \tilde{\varphi}^s(u_{d_1}(t), \hat{z}(t), \hat{z}_{\tau_s}(t), z(t), z_{\tau_s}(t)) - \theta\Delta_\theta^{-1}KC\tilde{z}(t) - B\varepsilon_{\tau_1}(t) \qquad (4.17)
\end{aligned}$$

where $\varepsilon_{\tau_1}(t) = \varepsilon(t-\tau_1)$ and

$$\tilde{\varphi}^s(u_{d_1}(t), \hat{z}(t), \hat{z}_{\tau_s}(t), z(t), z_{\tau_s}(t)) = \varphi^s(u_{d_1}(t), \hat{z}(t), \hat{z}_{\tau_s}(t)) - \varphi^s(u_{d_1}(t), z(t), z_{\tau_s}(t)) \tag{4.18}$$

Now, set $\bar{z}(t) = \theta^{n-1}\Delta_\theta\tilde{z}(t)$ as the exponential error and considering the following equalities:

$$\Delta_\theta A\Delta_\theta^{-1} = \theta A, \quad C\Delta_\theta^{-1} = C \text{ and } \theta^{n-1}\Delta_\theta B = B. \tag{4.19}$$

Bearing in mind this, one can show that:

$$\begin{aligned}
\dot{\bar{z}}(t) &= \theta A\bar{z}(t) + \theta^{n-1}\Delta_\theta\tilde{\varphi}^s(u_{d_1}(t), \hat{z}(t), \hat{z}_{\tau_s}(t), z(t), z_{\tau_s}(t)) - \theta KC\bar{z}(t) - B\varepsilon_{\tau_1}(t) \\
&= \tilde{A}\bar{z}(t) + \theta^{q-1}\Delta_\theta\tilde{\varphi}^s(u_{d_1}(t), \hat{z}(t), \hat{z}_{\tau_s}(t), z(t), z_{\tau_s}(t)) - B\varepsilon_{\tau_1}(t) \qquad (4.20)
\end{aligned}$$

where the Hurwitz matrix $\tilde{A}$ is defined in (4.13).

By considering the following Lyapunov candidate function:

$$V(\bar{z}(t)) = \bar{z}^T(t)P\bar{z}(t) \tag{4.21}$$

where $P = P^T > 0$ is the SPD matrix given by (4.14). Now, one obtains the time derivative of (4.21) as follows:

$$\dot{V}(\bar{z}(t)) = 2\bar{z}^T(t)P\dot{\bar{z}}(t). \tag{4.22}$$

Now, combining equations (4.20) and (4.14) in (4.22), one gets:

$$\begin{aligned}\dot{V}(\bar{z}(t)) \leq & -\frac{2\mu}{\lambda_M}\theta V(\bar{z}(t)) + 2\bar{z}^T(t)P\theta^{n-1}\Delta_\theta\tilde{\varphi}^s(u_{d_1}(t), \hat{z}(t), \hat{z}_{\tau_s}(t), z(t), z_{\tau_s}(t)) \\ & + 2\bar{z}^T(t)PB\varepsilon_{\tau_1}(t)\end{aligned} \tag{4.23}$$

where λ_M is the largest eigenvalue of matrix P.

Using the Lipschitz assumption on the nonlinearity $\tilde{\varphi}^{(s)}$ together with its triangular structure, one can show that

$$\begin{aligned}2\bar{z}^T(t)P\theta^{n-1}\Delta_\theta\tilde{\varphi}^s(u_{d_1}(t), \hat{z}(t), \hat{z}_{\tau_s}(t), z(t), z_{\tau_s}(t)) &\leq \alpha_1 V(\bar{z}(t)) + \alpha_2\|\bar{z}(t)\|\|\bar{z}_{\tau_s}(t)\| \\ &\leq c_1 V(\bar{z}(t)) + c_2 V(\bar{z}_{\tau_s}(t))\end{aligned} \tag{4.24}$$

where α_1 and α_2 are positive constants independent of θ and $c_1 = \alpha_1 + \alpha_2/(2\lambda_m)$, $c_2 = \alpha_2/2\lambda_m$ with λ_m being the smallest eigenvalue of P.

Combining (4.22) and (4.24) and choosing θ sufficiently high, in particular to satisfies $-\frac{2\mu}{\lambda_M}\theta + c_1 \leq -\frac{\mu}{\lambda_M}\theta$, leads to:

$$\dot{V}(\bar{z}(t)) \leq \frac{-\mu}{\lambda_M}\theta V(\bar{z}(t)) + c_2 V(\bar{z}_{\tau_s}(t)) + 2\sqrt{\lambda_M}\sqrt{V(\bar{z}(t))}|\varepsilon_{\tau_1}(t)|, \tag{4.25}$$

or equivalently, leads to

$$\frac{d}{dt}\sqrt{V(\bar{z}(t))} \leq \frac{-\mu}{2\lambda_M}\theta\sqrt{V(\bar{z}(t))} + \frac{c_2}{2}\sqrt{V(\bar{z}_{\tau_s}(t))} + 2\sqrt{\lambda_M}|\varepsilon_{\tau_1}(t)|. \tag{4.26}$$

According to Assumption **A4**, $\varepsilon_{\tau_1}(t)$ is essentially bounded with δ_ε as essential bound. Hence, the function $\sqrt{V(\bar{z}(t))}$ satisfies the conditions of lemma B.3.1 with

$$\delta = \tau_s, \quad a = \frac{\mu}{2\lambda_M}, \quad b = \frac{c_2}{2} \text{ and } p(t) = \sqrt{\lambda_M}|\varepsilon_{\tau_1}(t)|. \tag{4.27}$$

According to this lemma, one gets:

$$\sqrt{V(\bar{z}(t))} \leq e^{-\frac{ln(\theta)}{2\tau_s}t}\left(1 + 2b\tau_s\frac{(\sqrt{\theta}-1)}{ln(\theta)}\right)\max_{s\in[-\tau_s,0]}\sqrt{V(\bar{z}(s))} + \frac{2\tau_s\delta_\varepsilon}{ln(\theta)}. \tag{4.28}$$

where b is as given in (4.27).Coming back to $\bar{z}$, one gets:

$$\|\bar{z}(t)\| \leq \sigma_P e^{-\frac{ln(\theta)}{2\tau_s}t}\left(1+2b\tau_s\frac{(\sqrt{\theta}-1)}{ln(\theta)}\right)\max_{s\in[-\tau_s,0]}\|\bar{z}(s)\| + \frac{2\tau_s\delta_\varepsilon}{ln(\theta)}. \tag{4.29}$$

According to the definition $\bar{z}$, one has for $\theta \geq 1$ and for all $t \geq 0$,

$$\|\tilde{z}(t)\| \leq \|\bar{z}(t)\| \leq \theta^{q-1}\|\tilde{z}(t)\|$$

$$\|\tilde{z}(t)\| \leq \mu(\theta)e^{-\lambda(\theta)t}\max_{s\in[-\tau_s,0]}\|\tilde{z}(s)\| + M(\theta)\delta_\varepsilon \tag{4.30}$$

with

$$\lambda(\theta) = \frac{ln(\theta)}{2\tau_s},\ \mu(\theta) = \sigma_P\left(1+2b\tau_s\frac{(\sqrt{\theta}-1)}{ln(\theta)}\right) \text{ and } M(\theta) = 2\sigma_P\frac{\tau_s}{ln(\theta)}. \tag{4.31}$$

This concludes the proof of Theorem 4.3.1. ■

4.4 Design of the cascade predictors

As mentioned before, the cascade structure is considered for the estimation of the actual state vector. The first subsystem is a high gain observer that provides an accurate estimation of the delayed state of the system, i.e $\{x(t-\tau_1)\}_{t\geq 0}$ from the available delayed output measurements, i.e. $\{y_a(t)\}_{t\geq 0}$. The remaining m subsystems are the predictors that achieve estimation over a prediction horizon of τ_1/m. Now, one proceeds to define the following state variables and inputs for $j = 0, \ldots, m$,

$$x^{(j)}(t) = x\left(t-\tau_1+\frac{j}{m}\tau_1\right), u^{(j)}(t) = u\left(t-\tau_1+\frac{j}{m}\tau_1\right). \tag{4.32}$$

It should be emphasized that the main idea behind the introduction of the above variables $x^{(j)}$ and $u^{(j)}$ lies in the following properties satisfied by these variables

$$x^{(j)}\left(t-\frac{\tau_1}{m}\right) = x^{(j-1)}(t) \text{ and } u^{(j)}\left(t-\frac{\tau_1}{m}\right) = u^{(j-1)}(t), j = 1, \ldots, m. \tag{4.33}$$

The above properties (4.33) can be expressed as follows:

$$x^{(j)}(t) = x^{(j-1)}\left(t+\frac{\tau_1}{m}\right) \text{ and } u^{(j)}(t) = u^{(j-1)}\left(t+\frac{\tau_1}{m}\right), j = 1, \ldots, m. \tag{4.34}$$

The equality (4.34) implies the following:

$$x^{(j-1)}(t-\tau_1) = x^{(j-1)}\left(t-\tau_1+\frac{1}{m}\tau_1\right). \tag{4.35}$$

Using the above equality, the system (4.10) can be rewritten as follows:

$$\dot{x}^{(j)}(t) = Ax^{(j)}(t) + \varphi^{(s)}\left(u^{(j)}(t), x^{(j)}(t), x^{(j-1)}_{\tau_s}\left(t+\frac{\tau_1}{m}\right)\right) \tag{4.36}$$

according to the adopted notation, one has

$$x_{\tau_s}^{(j-1)}\left(t+\frac{\tau_1}{m}\right) = x_{\tau_s}\left(t+(j-1)\frac{\tau_1}{m}+j\frac{\tau_1}{m}\right) = x_{\tau_s}\left(t+j\frac{\tau_1}{m}\right) = x\left(t-\tau_s+j\frac{\tau_1}{m}\right) = x_{\tau_s}^{(j)}(t) \tag{4.37}$$

According to (4.32), one has $x^{(0)} = x(t-\tau_1) = x_{\tau_1}(t)$ and hence an estimate $\hat{x}^{(0)}$ of $x^{(0)}$ is already available through the high gain observer in the head (4.11).

4.5 Equations of the cascade predictors

Let us propose a predictor $\hat{x}^{(j)}$ that shall provide an estimate of the state $x^{(j)}, j = 1, \ldots m$. Miming structure of the high gain observer (4.11), the predictor $\hat{x}^{(j)}$ is given by the following equation:

$$\begin{cases} \dot{\hat{x}}^{(j)}(t) & = A\hat{x}^{(j)}(t) + \varphi^{(s)}\left(u^{(j)}(t), \hat{x}^{(j)}(t), \hat{x}_{\tau_s}^{(j-1)}\left(t+\frac{\tau_1}{m}\right)\right) - G^{(j)}(t), \\ G^{(j)}(t) & = e^{\bar{A}\frac{\tau_1}{m}}\left\{(A-\bar{A})\left(\hat{x}^{(j)}\left(t-\frac{\tau_1}{m}\right) - x^{(j-1)}(t)\right)\right. \\ & \quad \left. +\varphi^{(s)}\left(u^{(j-1)}(t), \hat{x}^{(j)}(t-\frac{\tau_1}{m}), \hat{x}_{\tau_s}^{(j-1)}(t)\right) - \varphi^{(s)}\left(u^{(j-1)}(t), \hat{x}^{(j-1)}(t), \hat{x}_{\tau_s}^{(j-1)}(t)\right)\right\} \end{cases} \tag{4.38}$$

where $\bar{A}$ is a Hurwitz matrix and m is a positive integer integer the choice of which shall be specified later. Recall that the implementation of the above predictors requires the availability of $\hat{x}_{\tau_s}^{(j-1)}\left(t+\frac{\tau_1}{m}\right) \triangleq \hat{x}_{\tau_s}^{(j-1)}\left(t-(\tau_s-\frac{\tau_1}{m})\right)$ at the actual time instant t. Such a condition is satisfied as soon as m is chosen such that $(\tau_s - \frac{\tau_1}{m}) \geq 0$, i.e $m \geq \frac{\tau_1}{\tau_s}$.

To summarize, the equations of the overall cascade observer constituted by the high gain observers at the head of the cascade and m successive predictors are:

$$\begin{cases} \dot{\hat{x}}^{(0)}(t) & = A\hat{x}^{(0)}(t) + \varphi^{(s)}\left(u^{(0)}(t), \hat{x}^{(0)}(t), \hat{x}_{\tau_s}^{(0)}(t)\right) - G^{(0)}(t), \\ \dot{\hat{x}}^{(j)}(t) & = A\hat{x}^{(j)}(t) + \varphi^{(s)}\left(u^{(j)}(t), \hat{x}^{(j)}(t), \hat{x}_{\tau_s}^{(j-1)}\left(t+\frac{\tau_1}{m}\right)\right) - G^{(j)}(t), \\ G^{(0)}(t) & = \theta\Delta_\theta^{-1}K\left(C\hat{x}^{(0)} - y_a(t)\right) \\ \text{for } j & = 1\ldots, m, \\ G^{(j)}(t) & = e^{\bar{A}\frac{\tau_1}{m}}\left\{(A-\bar{A})\left(\hat{x}^{(j)}\left(t-\frac{\tau_1}{m}\right) - x^{(j-1)}(t)\right)\right. \\ & \quad \left. +\varphi^{(s)}\left(u^{(j-1)}(t), \hat{x}^{(j)}(t-\frac{\tau_1}{m}), \hat{x}_{\tau_s}^{(j-1)}(t)\right) - \varphi^{(s)}\left(u^{(j-1)}(t), \hat{x}^{(j-1)}(t), \hat{x}_{\tau_s}^{(j-1)}(t)\right)\right\} \\ \hat{x}^{(j)}(s) & = \phi\left(s-\tau_s+\frac{j}{m}\tau_1\right), \forall s \in [-\tau_s - \frac{j}{m}\tau_1], j = 0, \ldots, m. \end{cases} \tag{4.39}$$

where $\phi(s)$ is an arbitrary estimate of $x(s)$ for $s \in [-\tau_s - \frac{j}{m}\tau_1, 0]$.

Before giving the main theorem that summarizes the results obtained through the above developments, one recalls that since the matrix $\bar{A}$ is Hurwitz, there exists a positive number $\beta \geq 1$ such that Farza et al. (2018a)

$$\forall t \geq 0 : \|e^{\bar{A}t}\| \leq \beta e^{-\bar{a}t}, \tag{4.40}$$

with $\bar{a} = \min_{i\in\{1,\ldots,n\}} |\Re(\lambda_i(\bar{A}))|$ where $\lambda_i(\bar{A}), i = 1, \ldots, n$ are the n eigenvalues of $\bar{A}$ (with negative real parts).

Now, one can propose the following theorem.

Theorem 4.5.1 *Consider system (4.10) subject to Assumptions **A1** to **A4** and observer (4.39). If the matrix $\bar{A}$ is chosen such that $2\bar{a} \leq \lambda(\theta)$ where $\bar{a}$ and $\lambda(\theta) > 0$ are respectively defined as in (4.40) and (4.16) and if the number m is selected such that*

$$\frac{\tau_1}{m} \leq \tau_s \text{ and } \eta\frac{\tau_1}{m} < 1, \text{ with } \eta = \beta(L_\varphi + \|\bar{A} - A\|), \tag{4.41}$$

*where β is given by (4.40) and L_φ is the Lipschitz constant of φ defined in **A2** then, one has for $j = 1, \ldots, m$,*

$$\forall t \geq 0, \|\tilde{x}^{(j)}(t)\| \leq \rho_j e^{-\frac{\bar{a}}{2}t} + M_j\delta_\varepsilon, \tag{4.42}$$

where ρ_j and M_j, $j = 1, \ldots, m$ satisfies the following recursive equations

$$\rho_j = \frac{2}{\bar{a}}\frac{(\beta L_\varphi e^{\frac{\bar{a}}{2}(\tau_s - \frac{\tau_1}{m})} + \eta e^{-\bar{a}\frac{\tau_1}{m}})}{1 - \eta\frac{\tau_1}{m}}\rho_{j-1} + \frac{1}{1 - \eta\frac{\tau_1}{m}}\left(\beta\|\tilde{x}^{(j)}(0)\| + 2\eta\int_{-\frac{\tau_1}{m}}^{0}\|\tilde{x}^{(j)}(s)\|ds\right), \tag{4.43}$$

$$M_j = \frac{1}{\bar{a}}\frac{\left(\beta L_\varphi + \eta e^{-\bar{a}\frac{\tau_1}{m}}\right)}{1 - \eta\frac{\tau_1}{m}}M_{j-1} + \frac{\beta}{\bar{a}\left(1 - \eta\frac{\tau_1}{m}\right)}. \tag{4.44}$$

*and $\rho_0 = \mu(\theta)\max_{s\in[-\tau_s,0]}\|\tilde{x}^{(0)}(s)\|$, $M_0 = M(\theta)$ where $\mu(\theta)$ and $M(\theta)$ are given by (4.16), and δ_ε is the upper essential bound of the uncertainties as given by assumption **A4**.*

4.5.1 Proof of Theorem 4.5.1

Let us define the observation error of each predictor as $\tilde{x}^{(j)} = \hat{x}^{(j)} - x^{(j)}$ for $j = 1, \ldots, m$. From (4.36) and (4.38), one obtains for $j = 1, \ldots, m$,

$$\begin{aligned}
\dot{\tilde{x}}^{(j)}(t) =& A\tilde{x}^{(j)} + \varphi^{(s)}\left(u^{(j)}(t), \hat{x}^{(j)}(t), \hat{x}^{(j-1)}_{\tau_s}\left(t + \frac{\tau_1}{m}\right)\right) - \varphi^{(s)}\left(u^{(j)}(t), x^{(j)}(t), x^{(j-1)}_{\tau_s}\left(t + \frac{\tau_1}{m}\right)\right) \\
& - e^{\bar{A}\frac{\tau_1}{m}}\left\{(A - \bar{A})\left(\hat{x}^{(j)}\left(t - \frac{\tau_1}{m}\right) - \hat{x}^{(j-1)}(t)\right)\right. \\
& \left. + \varphi^{(s)}\left(u^{(j-1)}(t), \hat{x}^{(j)}\left(t - \frac{\tau_1}{m}\right), \hat{x}^{(j-1)}_{\tau_s}(t)\right) - \varphi^{(s)}\left(u^{(j-1)}(t), \hat{x}^{(j-1)}(t), \hat{x}^{(j-1)}_{\tau_s}(t)\right)\right\} \\
& - B\varepsilon^{(j)}(t).
\end{aligned} \tag{4.45}$$

Now, one defines $\zeta^{(j)}(t)$ as follows:

$$\begin{aligned}
\zeta^{(j)}(t) =& \tilde{x}^{(j)}(t) - \int_{t-\frac{\tau_1}{m}}^{t} e^{\bar{A}(t-s)}\left\{\varphi^{(s)}\left(u^{(j)}(s), \hat{x}^{(j)}(s), \hat{x}^{(j-1)}_{\tau_s}\left(s + \frac{\tau_1}{m}\right)\right) - \right. \\
& \left. \varphi^{(s)}\left(u^{(j)}(s), x^{(j)}(s), x^{(j-1)}_{\tau_s}\left(s + \frac{\tau_1}{m}\right)\right) + (A - \bar{A})\tilde{x}^{(j)}(s)\right\} ds.
\end{aligned} \tag{4.46}$$

One gets

$$\begin{aligned}
\dot{\zeta}^{(j)}(t) =& \dot{\tilde{x}}^{(j)}(t) - \bar{A}(\tilde{x}^{(j)}(t) - \zeta^{(j)}(t)) - \varphi^{(s)}\left(u^{(j)}(s), \hat{x}^{(j)}(s), \hat{x}^{(j-1)}_{\tau_s}\left(s + \frac{\tau_1}{m}\right)\right) \\
& + \varphi^{(s)}\left(u^{(j)}(s), x^{(j)}(s), x^{(j-1)}_{\tau_s}\left(s + \frac{\tau_1}{m}\right)\right) - (A - \bar{A})\tilde{x}^{(j)}(t) \\
& + e^{\bar{A}\frac{\tau_1}{m}}\left\{\varphi^{(s)}\left(u^{(j-1)}(t), \hat{x}^{(j)}\left(t - \frac{\tau_1}{m}\right), \hat{x}^{(j-1)}_{\tau_s}(t)\right)\right. \\
& \left. - \varphi^{(s)}\left(u^{(j-1)}(t), \hat{x}^{(j-1)}(t), \hat{x}^{(j-1)}_{\tau_s}(t)\right) + (A - \bar{A})\tilde{x}^{(j)}\left(t - \frac{\tau_1}{m}\right)\right\} \\
=& A\zeta^{(j)}(t) + \dot{\tilde{x}}^{(j)} - A\tilde{x}^{(j)}(t) \\
& - \left\{\varphi^{(s)}\left(u^{(j)}(t), \hat{x}^{(j)}(t), \hat{x}^{(j-1)}_{\tau_s}\left(t + \frac{\tau_1}{m}\right)\right) - \varphi^{(s)}\left(u^{(j)}(t), x^{(j)}(t), x^{(j-1)}_{\tau_s}\left(t + \frac{\tau_1}{m}\right)\right)\right\} \\
& + e^{\bar{A}\frac{\tau_1}{m}}\left\{\varphi^{(s)}\left(u^{(j-1)}(t), x^{(j)}\left(t - \frac{\tau_1}{m}\right), x^{(j-1)}_{\tau_s}(t)\right) - \varphi^{(s)}\left(u^{(j-1)}(t), x^{(j-1)}(t), x^{(j-1)}_{\tau_s}(t)\right)\right. \\
& \left. + (A - \bar{A})\left(\hat{x}^{(j)}\left(t - \frac{\tau_1}{m}\right) - \hat{x}^{(j-1)}(t)\right)\right\}
\end{aligned} \tag{4.47}$$

Substituting in the above last equality, $\dot{\tilde{x}}^{(j)}(t)$, by its expression (4.45), one gets

$$\begin{aligned}
\dot{\zeta}^{(j)}(t) =& \bar{A}\zeta^{(j)}(t) + \varphi^{(s)}\left(u^{(j)}(t), \hat{x}^{(j)}(t), \hat{x}^{(j-1)}_{\tau_s}\left(t + \frac{\tau_1}{m}\right)\right) - \varphi^{(s)}\left(u^{(j)}(t), x^{(j)}(t), x^{(j-1)}_{\tau_s}\left(t + \frac{\tau_1}{m}\right)\right) \\
& + B\varepsilon^{(j)}(t) + e^{\bar{A}\frac{\tau_1}{m}}\left\{(A - \bar{A})\tilde{x}^{(j-1)}(t) + \varphi^{(s)}\left(u^{(j-1)}(t), \hat{x}^{(j-1)}(t), \hat{x}^{(j-1)}_{\tau_s}(t)\right)\right. \\
& \left. - \varphi^{(s)}\left(u^{(j-1)}(t), x^{(j-1)}(t), x^{(j-1)}_{\tau_s}(t)\right)\right\}.
\end{aligned} \tag{4.48}$$

The state $\zeta^{(j)}(t)$ governed by the above ODE can be expressed as follows

$$\begin{aligned}\zeta^{(j)}(t) =& e^{\bar{A}t}\zeta^{(j)}(0) + \int_0^t e^{\bar{A}(t-s)}\left\{B\varepsilon^{(j)}(s) + \varphi^{(s)}\left(u^{(j)}(s), \hat{x}^{(j)}(s), \hat{x}_{\tau_s}^{(j-1)}\left(s+\frac{\tau_1}{m}\right)\right)\right.\\ &\left.-\varphi^{(s)}\left(u^{(j)}(s), x^{(j)}(s), x_{\tau_s}^{(j-1)}\left(s+\frac{\tau_1}{m}\right)\right)\right\}ds\\ &+\int_0^t e^{\bar{A}\left(t-s+\frac{\tau_1}{m}\right)}\left\{(A-\bar{A})\tilde{x}^{(j-1)}(s) + \varphi^{(s)}\left(u^{(j-1)}(s), \hat{x}^{(j-1)}(s), \hat{x}_{\tau_s}^{(j-1)}(s)\right)\right.\\ &\left.-\varphi^{(s)}\left(u^{(j-1)}(s), x^{(j-1)}(s), x_{\tau_s}^{(j-1)}(s)\right)\right\}ds \qquad (4.49)\end{aligned}$$

Using (4.40) and the Lipschitz property (4.5) of the nonlinearity $\varphi^{(s)}$, the state $\zeta^{(j)}$ expressed by (4.49) can be bounded as follows:

$$\begin{aligned}\|\zeta^{(j)}(t)\| \leq& \|e^{\bar{A}t}\zeta^{(j)}(0)\| + \beta\int_0^t e^{-\bar{a}(t-s)}\left(|\varepsilon^{(j)}(s)| + L_\varphi\|\tilde{x}^{(j-1)}\left(s+\frac{\tau_1}{m}\right)\|\right)ds\\ &+\beta\int_0^t e^{-\bar{a}(t-s+\frac{\tau_1}{m})}\left(\|A-\bar{A}\| + L_\varphi\right)\|\tilde{x}^{(j-1)}(s)\|ds\\ =&\|e^{\bar{A}t}\zeta^{(j)}(0)\| + \beta\int_0^t e^{-\bar{a}(t-s)}\left(|\varepsilon^{(j)}(s)| + L_\varphi\|\tilde{x}^{(j-1)}\left(s+\frac{\tau_1}{m}\right)\|\right)ds\\ &+\eta\int_0^t e^{-\bar{a}(t-s+\frac{\tau_1}{m})})\|\tilde{x}^{(j-1)}(s)\|ds \qquad (4.50)\end{aligned}$$

Using the induction assumption, one gets

$$\begin{aligned}\|\zeta^{(j)}(t)\| \leq& \|e^{\bar{A}t}\zeta^{(j)}(0)\| + \beta\int_0^t e^{-\bar{a}(t-s)}\left(|\varepsilon^{(j)}(s)| + L_\varphi\left(\rho_{j-1}e^{-\frac{\bar{a}}{2}(s-\tau_s+\frac{\tau_1}{m})}M_{j-1}\delta_\varepsilon\right)\right)ds\\ &+\eta\int_0^t e^{-\bar{a}(t-s+\frac{\tau_1}{m})}(\rho_{j-1}e^{-\frac{\bar{a}}{2}s} + M_{j-1}\delta_\varepsilon)ds\\ =&\|e^{\bar{A}t}\zeta^{(j)}(0)\| + \int_0^t e^{-\bar{a}(t-s)}\left(\beta|\varepsilon^{(j)}(s)| + \left(\beta L_\varphi + \eta e^{-\bar{a}\frac{\tau_1}{m}}\right)M_{j-1}\delta_\varepsilon\right)ds\\ &+e^{\bar{a}t}\int_0^t\left(\beta L_\varphi e^{\frac{\bar{a}}{2}\left(\tau_s-\frac{\tau_1}{m}\right)} + \eta e^{-\bar{a}\frac{\tau_1}{m}}\right)\rho_{j-1}e^{\frac{\bar{a}}{2}s}ds\\ \leq&\|e^{\bar{A}t}\zeta^{(j)}(0)\| + \frac{1}{a}\left(\beta\delta_\varepsilon + \left(\beta L_\varphi + \eta e^{-\bar{a}\frac{\tau_1}{m}}\right)M_{j-1}\delta_\varepsilon\right)\\ &+\frac{2}{\bar{a}}\left(\beta L_\varphi e^{\frac{\bar{a}}{2}\left(\tau_s-\frac{\tau_1}{m}\right)} + \eta e^{-\bar{a}\frac{\tau_1}{m}}\right)\rho_{j-1}e^{-\frac{\tau_1}{m}t} \qquad (4.51)\end{aligned}$$

According to (4.46), one has

$$\begin{aligned}e^{\bar{A}t}\zeta^{(j)}(0) =& e^{\bar{A}t}\tilde{x}^{(j)}(0) - \int_{-\frac{\tau_1}{m}}^0 e^{\bar{A}(t-s)}\left\{\varphi^{(s)}\left(u^{(j)}(s), \hat{x}^{(j)}(s), \hat{x}_{\tau_s}^{(j-1)}\left(s+\frac{\tau_1}{m}\right)\right)\right.\\ &\left.\varphi^{(s)}\left(u^{(j)}(s), x^{(j)}(s), x_{\tau_s}^{(j-1)}\left(s+\frac{\tau_1}{m}\right)\right) + (A-\bar{A})\tilde{x}^{(j)}(s)\right\}ds \qquad (4.52)\end{aligned}$$

and hence

$$\|e^{\bar{A}t}\zeta^{(j)}(0)\| \leq \beta e^{-\bar{a}t}\left(\|\tilde{x}^{(j)}(0)\| + \eta\int_{-\frac{\tau_1}{m}}^{0}\|\tilde{x}^{(j)}(s)\|ds\right),$$
$$\leq \beta e^{-\frac{\bar{a}}{2}t}\left(\|\tilde{x}^{(j)}(0)\| + \eta\int_{-\frac{\tau_1}{m}}^{0}\|\tilde{x}^{(j)}(s)\|ds\right). \tag{4.53}$$

Combining the last inequality with (4.52), one gets

$$\|\zeta^{(j)}(t)\| \leq \frac{1}{\bar{a}}\left(\beta + \left(\beta L_\varphi + \eta e^{-\bar{a}\frac{\tau_1}{m}}\right)M_{j-1}\right)\delta_\varepsilon$$
$$+\left(\beta\|\tilde{x}^{(j)}(0)\| + \eta\int_{-\frac{\tau_1}{m}}^{0}\|\tilde{x}^{(j)}(s)\|ds + \frac{2}{\bar{a}}\left(\beta L_\varphi e^{\frac{\bar{a}}{2}\left(\tau_s - \frac{\tau_1}{m}\right)} + \eta e^{-\bar{a}\frac{\tau_1}{m}}\right)\rho_{j-1}\right)e^{-\frac{\bar{a}}{2}t} \tag{4.54}$$

Now, from (4.46), one gets

$$\|\tilde{x}^{(j)}(t)\| \leq \|\zeta^{(j)}(t)\| + \int_{t-\frac{\tau_1}{m}}^{t} e^{\bar{A}(t-s)}\left\{\varphi^{(s)}\left(u^{(j)}(s), \hat{x}^{(j)}(s), \hat{x}_{\tau_s}^{(j-1)}\left(s+\frac{\tau_1}{m}\right)\right) - \right.$$
$$\left.\varphi^{(s)}\left(u^{(j)}(s), x^{(j)}(s), x_{\tau_s}^{(j-1)}\left(s+\frac{\tau_1}{m}\right)\right) + (A-\bar{A})\tilde{x}^{(j)}(s)\right\}ds$$
$$\leq \|\zeta^{(j)}(t)\| + \beta\int_{t-\frac{\tau_1}{m}}^{t} e^{-\bar{a}(t-s)}\left(L_\varphi + (A-\bar{A})\right)\|\tilde{x}^{(j)}(s)\|ds$$
$$= \|\zeta^{(j)}(t)\| + \eta\int_{t-\frac{\tau_1}{m}}^{t} e^{-\bar{a}(t-s)}\|\tilde{x}^{(j)}(s)\|ds \tag{4.55}$$

Using the bound of $\|\zeta^{(j)}(t)\|$ given by (4.53), the last inequality leads to

$$\|\tilde{x}^{(j)}(t)\| \leq \left(\beta\|\tilde{x}^{(j)}(0)\| + \eta\int_{-\frac{\tau_1}{m}}^{0}\|\tilde{x}^{(j)}(s)\|ds + \frac{2}{\bar{a}}\left(\beta L_\varphi e^{\frac{\bar{a}}{2}\left(\tau_s - \frac{\tau_1}{m}\right)} + \eta e^{-\bar{a}\frac{\tau_1}{m}}\right)\rho_{j-1}\right)e^{-\frac{\bar{a}}{2}t}$$
$$+\frac{1}{\bar{a}}\left(\beta + \left(\beta L_\varphi + \eta e^{-\bar{a}\frac{\tau_1}{m}}\right)M_{j-1}\right)\delta_\varepsilon + \eta\int_{t-\frac{\tau_1}{m}}^{t} e^{-\bar{a}(t-s)}\|\tilde{x}^{(j)}(s)\|ds$$
$$\triangleq \rho e^{-\frac{\bar{a}}{2}t} + M + \eta\int_{t-\frac{\tau_1}{m}}^{t} e^{-\bar{a}(t-s)}\|\tilde{x}^{(j)}(s)\|ds$$
$$\leq \rho e^{-\frac{\bar{a}}{2}t} + M + \eta\int_{t-\frac{\tau_1}{m}}^{t} e^{-\bar{a}(t-s)}\|\tilde{x}^{(j)}(s)\|ds \tag{4.56}$$

with

$$\rho = \beta\|\tilde{x}^{(j)}(0)\| + \eta\int_{-\frac{\tau_1}{m}}^{0}\|\tilde{x}^{(j)}(s)\|ds + \frac{2}{\bar{a}}\left(\beta L_\varphi e^{\frac{\bar{a}}{2}\left(\tau_s - \frac{\tau_1}{m}\right)} + \eta e^{-\bar{a}\frac{\tau_1}{m}}\right)\rho_{j-1}$$
$$M = \frac{1}{\bar{a}}\left(\beta + \left(\beta L_\varphi + \eta e^{-\bar{a}\frac{\tau_1}{m}}\right)M_{j-1}\right)\delta_\varepsilon \tag{4.57}$$

Since $\eta\frac{\tau_1}{m} < 1$, all conditions of lemma B.3.2 are satisfied and one gets

$$\|\tilde{x}^{(j)}(t)\| \leq \frac{1}{1-\eta\frac{\tau_1}{m}}\left(\left(\rho + \eta\int_{-\frac{\tau_1}{m}}^{0}\|\tilde{x}^{(j)}(s)\|ds\right)e^{-\frac{\bar{a}}{2}t} + M\right) \triangleq \rho_j e^{-\frac{\bar{a}}{2}t} + M_j\delta_\varepsilon \qquad (4.58)$$

Substituting in the last equality ρ and M by their respective expressions given by (4.57) leads to inequality (4.42). This ends the proof of Theorem 4.5.1. ■

4.6 Extension of the predictor design

One of the main constraints of the system (4.10), subject to the design of the predictors, is that the presence of the delayed state is allowed only in the triangular function and in the output. However, in this section, we will extend the design by considering the delayed state at the end of the state. Let us consider the following system:

$$\mathcal{SYS}: \begin{cases} \dot{x}(t) &= Ax_\tau(t) + \varphi^s(u(t), x(t), x_{\tau_s}(t)) + B\varepsilon(t) \\ y_a(t) &= x_1(t-\tau_1) \end{cases} \qquad (4.59)$$

where

$$x_\tau(t) = \begin{bmatrix} x_1(t-\tau_1) \\ x_2(t-\tau_2) \\ \vdots \\ x_n(t-\tau_n) \end{bmatrix} \qquad (4.60)$$

with $\tau_i \geq 0, i = 1, \ldots, n$ and $\tau \triangleq \sum_{i=1}^{n}\tau_i > 0$, and the other variables keep the same meaning as in system (4.10).

It is clear that system (4.59) is identical to system (4.10) in the case where $\tau_i = 0,\ i = 2, \ldots, n$ and hence it includes the latter it allows the presence of delayed states in a non triangular manner. Similarly to the output delay τ_1, the new state delays $\tau_i, i = 2, \ldots, n$ may be arbitrarily large. To extend the predictor design proposed in the above section, one introduces the following change of variables::

$$\forall t \geq 0, \xi_i(t) = x_i(t-\bar{\tau}_i), i = 1, \ldots, n, \qquad (4.61)$$

where

$$\bar{\tau}_i = \sum_{j=1}^{i}\tau_j, i = 1, \ldots, n. \qquad (4.62)$$

For clarity purposes, let us focus on the expression of the time derivative of $\xi_i, i = 1, \ldots, n$. To this end, one recalls that the time derivative of the original state $x_i(t)$ can be written in the following developed form

$$\dot{x}_i(t) = x_{i+1}(t - \tau_{i+1}) + \varphi^{(s)}\left(u(t), x_1(t), \ldots, x_i(t), x_1(t - \tau_s), \ldots, x_i(t - \tau_s)\right). \qquad (4.63)$$

From the definition of $\xi(t)$ given by (4.61), one gets for $i = 1, \ldots, n$,

$$\begin{aligned}
\dot{\xi}_i(t) =& x_{i+1}(t - \bar{\tau}_i - \tau_{i+1}) \\
&+ \varphi^{(s)}(u(t - \bar{\tau}_i), x_1(t - \bar{\tau}_i), \ldots, x_i(t - \bar{\tau}_i), x_1(t - \tau_s - \bar{\tau}_i), \ldots, x_i(t - \tau_s - \bar{\tau}_i)) \\
=& \xi_{i+1}(t) + \varphi^{(s)}\left(u(t - \bar{\tau}_i), \xi_1(t - (\tau_2 + \ldots + \tau_i)), \ldots, \xi_{i-1}(t - \tau_i), \xi_i(t),\right. \\
&\left.\xi_1(t - (\tau_s + \tau_2 + \ldots + \tau_i)), \ldots, \xi_{i-1}(t - (\tau_s + \tau_i)), \xi_i(t - \tau_s)\right) \\
=& \xi_{i+1}(t) + \varphi^{(s)}\left(u(t - \bar{\tau}_i), \xi_1(t - (\bar{\tau}_i - \bar{\tau}_1)), \ldots, \xi_{i-1}(t - (\bar{\tau}_i - \bar{\tau}_{i-1})), \xi_i(t),\right. \\
&\left.\xi_1(t - (\tau_s + \bar{\tau}_i - \bar{\tau}_{i-1})), \ldots, \xi_{i-1}(t - (\tau_s + \bar{\tau}_i - \bar{\tau}_{i-1})), \xi_i(t - \tau_s)\right) \qquad (4.64)
\end{aligned}$$

For writing compactness, one introduces the following notations:

$$\bar{\tau}_{(i,j)} = \bar{\tau}_i - \bar{\tau}_j, i = 2, \ldots, n \text{ and } j = 1, \ldots, i - 1, \qquad (4.65)$$

and $\xi_{i,d}(t) \triangleq \xi_i(t - d)$ for any positive real d.

Using the above notations, equation (4.64) can be written as follows

$$\begin{aligned}
\dot{\xi}_i(t) =& \xi_{i+1}(t) + \varphi_i^{(s)}\Big(u_{\bar{\tau}_i}(t), \xi_{1,\bar{\tau}_{(i,1)}}(t), \ldots, \xi_{i-1,\bar{\tau}_{(i,i-1)}}(t), \xi_i(t), \\
&\xi_{i,\tau_s+\bar{\tau}_{(i,1)}}(t), \ldots, \xi_{i-1,\tau_s+\bar{\tau}_{(i,i-1)}}(t), \xi_{i,\tau_s}(t)\Big) \qquad (4.66)
\end{aligned}$$

For writing compactness, one shall bring together all the delayed versions of the components $\xi_i, i = 1, \ldots, n$ in one vector denoted ξ_τ, i.e.

$$\xi_\tau(t) \triangleq \begin{bmatrix} \xi_{1,\tau}(t) \\ \xi_{2,\tau}(t) \\ \vdots \\ \xi_{n-1,\tau}(t) \\ \xi_{n,\tau}(t) \end{bmatrix} \text{ with } \xi_{i,\tau}(t) \triangleq \begin{bmatrix} \xi_{i,\bar{\tau}_{(i+1,i)}}(t) \\ \vdots \\ \xi_{i,\bar{\tau}_{(i+1,n)}}(t) \\ \xi_{i,\tau_s+\bar{\tau}_{(i+1,i)}}(t) \\ \vdots \\ \xi_{i,\tau_s+\bar{\tau}_{(i+1,n)}}(t) \end{bmatrix}, i = 1, \ldots, n-1 \text{ and } \xi_{n,\tau}(t) \triangleq \xi_{n,\tau_s}(t). \qquad (4.67)$$

Using the above compact notation together with (4.66), one can check that the change of variables (4.61) puts system (4.59) under the following form

$$\mathcal{SYS} : \begin{cases} \dot{\xi}(t) &= A\xi(t) + \varphi^s(u_{\bar{\tau}_1}(t), \ldots, u_{\bar{\tau}_n}(t), \xi(t), \xi_{\tau_s}(t)) + B\varepsilon(t) \\ y_a(t) &= \xi_1(t) \end{cases} \qquad (4.68)$$

where $u_{\bar{\tau}_i}(t) = u(t - \bar{\tau}_i)$, and the nonlinearity, $\varphi^{(s)}$,keeps its triangular structure with respect to ξ and its delayed version ξ_τ.

Now with this it is necessary to propose a cascade to deal with this problem. Such a cascade will predict the state ξ over a prediction horizon equal to $\bar{\tau}_n$, i.e. to provide a prediction $\hat{\xi}(t+\bar{\tau}_n)$ of the estimate $\hat{\xi}(t)$ provided by the observer in the head. Indeed, since $\bar{\tau}_n \geq \bar{\tau}_i, i = 1, \ldots, n$, the availability of the prediction $\hat{\xi}(t + \bar{\tau}_n)$ implies that prediction $\xi(t + \bar{\tau}_i), i = 1, \ldots, n$ are also available and hence according to (4.61), the original state $x_i = \xi_i(t + \bar{\tau}_i)$ can be recovered.

4.6.1 Cascade observer redesign

Miming (4.32), one defines the states and inputs of the cascade observer as follows

$$\xi^{(j)}(t) = \xi\left(t + \frac{j}{m}\bar{\tau}_n\right), u_{\bar{\tau}_i}^{(j)}(t) = u_{\bar{\tau}_i}\left(t + \frac{j}{m}\bar{\tau}_n\right), j = 0, \ldots, m. \tag{4.69}$$

Notice that similarly to (4.34) and (4.35), the states $\xi^{(j)}$ satisfy the following equalities for $j = 1, \ldots, m$,

$$\xi^{(j)}(t) = \xi^{(j-1)}\left(t + \frac{\bar{\tau}_n}{m}\right), \xi^{(j)}(t - \tau_s) = \xi^{(j-1)}\left(t - \tau_s + \frac{\bar{\tau}_n}{m}\right) \tag{4.70}$$

The design of predictor shall be achieved on the basis of the following model, which is obtained from system (4.68) by miming the obtaining of system (4.36) from system (4.10)

$$\dot{\xi}^{(j)}(t) = A\xi^{(j)}(t) + \varphi^{(s)}\left(u_{\bar{\tau}_1}^{(j)}(t), \ldots, u_{\bar{\tau}_n}^{(j)}(t), \xi^{(j)}, \xi_\tau^{j-1}\left(t + \frac{\bar{\tau}_n}{m}\right)\right). \tag{4.71}$$

It should be emphasized that the number m of predictors in the cascade has to be chosen under a condition similar to (4.41) is satisfied. The underlying condition specializes as follows for the cascade observer (4.73),

$$\frac{\bar{\tau}_n}{m} \leq \min(\tau_s, \bar{\tau}_1, \ldots, \bar{\tau}_n) \text{ and } \eta\frac{\bar{\tau}_n}{m} < 1, \text{ with } \eta = \beta(L_\varphi + \|\bar{A} - A\|). \tag{4.72}$$

Now, miming (4.39), the equations of the cascade observer to predict $\xi(t + \bar{\tau}_n)$ are to be summarized as follows

$$\left\{\begin{array}{ll}\dot{\hat{\xi}}^{(0)}(t) & = A\hat{\xi}^{(0)}(t) + \varphi^{(s)}\left(u_{\bar{\tau}_1}(t),\ldots,u_{\bar{\tau}_n}(t),\hat{\xi}^{(0)}(t),\hat{\xi}^{(0)}_{\tau}(t)\right) - G^{(0)}(t),\\ \dot{\hat{\xi}}^{(j)}(t) & = A\hat{\xi}^{(j)}(t) + \varphi^{(s)}\left(u^{(j)}_{\bar{\tau}_1}(t),\ldots,u^{(j)}_{\bar{\tau}_n}(t),\hat{\xi}^{(j)}(t),\hat{\xi}^{(j-1)}_{\tau_s}\left(t+\frac{\bar{\tau}_n}{m}\right)\right) - G^{(j)}(t),\\ G^{(0)}(t) & = \theta\Delta_\theta^{-1}K\left(C\hat{\xi}^{(0)}(t) - y_a(t)\right)\\ \text{for } j & = 1\ldots,m,\\ G^{(j)}(t) & = e^{\bar{A}\frac{\bar{\tau}_n}{m}}\left\{(A-\bar{A})\left(\hat{\xi}^{(j)}\left(t-\frac{\bar{\tau}_n}{m}\right) - \xi^{(j-1)}(t)\right)\right.\\ & \quad +\varphi^{(s)}\left(u^{(j-1)}_{\bar{\tau}_1}(t),\ldots,u^{(j-1)}_{\bar{\tau}_n}(t),\hat{\xi}^{(j)}\left(t-\frac{\bar{\tau}_n}{m}\right),\hat{\xi}^{(j-1)}_{\tau_s}(t)\right)\\ & \quad \left.-\varphi^{(s)}\left(u^{(j-1)}_{\bar{\tau}_1}(t),\ldots,u^{(j-1)}_{\bar{\tau}_n}(t),\hat{\xi}^{(j-1)}(t),\hat{\xi}^{(j-1)}_{\tau_s}(t)\right)\right\}\\ \hat{\xi}^{(j)}(s) & = \phi\left(s-\tau_s+j\frac{\bar{\tau}_n}{m}\right), \forall s\in[-\tau_s - j\frac{\bar{\tau}_n}{m}], j=0,\ldots,m.\end{array}\right. \tag{4.73}$$

where $\phi(s)$ is an arbitrary estimate of $\xi(s)$ for $s \in [-\tau_s - \bar{\tau}_n, 0]$.

4.6.2 System state recovery

The objective is to recover the estimates of the actual state variables, i.e $\hat{x}_i(t)$ for $i = 1,\ldots,n$, of the original system state $x(t)$ from the observer design model state estimate sequence $\{\xi^{(j)}\}$ provided by the cascade observer (4.73) for $j = 0,\ldots,m$. For clarity purposes, two cases shall be considered depending on whether the state delays d_1 to d_n are commensurable or not.

Commensurable case

Let us consider the case where the output and state delays $\tau_i, i = 1,\ldots,n$ are commensurable, i.e.

$$\tau_i = m_i h, i = 1,\ldots,n, \tag{4.74}$$

where the $m_i, i = 1,\ldots,n$ are non negative integers and h is a positive real. For clarity purposes and without loss of generality, one assumes that the positive real h can be chosen arbitrarily small by appropriately scaling the integers m_i.

According to (4.74), one has

$$\bar{\tau}_i = \bar{m}_i h, \quad \bar{m}_i = \sum_{j=1}^{i} m_j, i = 1,\ldots,n. \tag{4.75}$$

Thus, after scaling h and the integers m_i in (4.74) if necessary, the number m of the predictors in the cascade observer (4.73) can be taken equal to

$$\bar{m}_n = \sum_{j=1}^{n} m_j. \tag{4.76}$$

According to the definition of the ξ_i's $i = 1, \ldots, n$ given by (4.61) and using (4.75), the original state components $x_i(t)$'s can be recovered from ξ_i's as follows

$$x_i(t) = \xi_i\left(t + \bar{\tau}_i\right) = \xi(t + \bar{m}_i h) = \xi_i^{\bar{m}_i}(t), \tag{4.77}$$

where the last equality results form the definition of the $\xi^{(j)}, j = 0, \ldots, m$ given by (4.61). Hence, the estimates of the original state components can be recovered from those of the cascade observer (4.73) as follows

$$\hat{x}_i(t) = \hat{\xi}_i^{\bar{m}_i}(t), i = 1, \ldots, m. \tag{4.78}$$

Non commensurable case

Now, one consider the case where the output and state delays $\tau_i, i = 1, \ldots, n$, are non commensurable. Let m be the number of predictors in the cascade and set

$$h = \frac{\bar{\tau}_n}{m} \tag{4.79}$$

where $\bar{\tau}_n$ is defined by (4.61) with $i = n$. Similarly to the case of commensurable delays, one has

$$\hat{x}_n(t) = \hat{\xi}_n^{(m)}(t). \tag{4.80}$$

It remains to provide estimates of $x_i(t)$, $i = 1, \ldots, n-1$. To this end, set $\bar{m}_i = \frac{\bar{\tau}_i}{h}$, $i = 1, \ldots, n-1$, where $\bar{\tau}_i$ is as defined in (4.61). According to the (4.61), one has

$$x_i(t) = \xi_i(t + \bar{\tau}_i) = \xi_i(t + \bar{m}_i h). \tag{4.81}$$

It should be emphasized that all the state components of the cascade observer (4.73) are predictors of $\xi(t)$ and underlying prediction horizon are multiple of the real h. However, since the τ_i's, $i = 1, \ldots, n$ are non commensurable, there exists at least one $i \in \{1, \ldots, n-1\}$ such that $\bar{m}_i$ is not an integer and hence the variable $\xi(t + \bar{\tau}_i h)$ is not a state of the cascade observer (4.73). Indeed, suppose that $\bar{m}_i$ is not an integer and let $\lfloor m_i \rfloor$ be the floor of $\bar{m}_i$ (the greatest integer which is lesser than $\bar{m}_i$). Then, $\hat{\xi}_i\left(t + \left(\lfloor \bar{m}_i \rfloor + 1\right) h\right)$ is indeed a state component of the cascade observer (4.73) and according to (4.70), one has

$$\hat{\xi}_i\left(t + \left(\lfloor \bar{m}_i \rfloor + 1\right) h\right) = \hat{\xi}_i^{(\lfloor \bar{m}_i \rfloor + 1)}(t). \tag{4.82}$$

Otherwise and according to the definition of ξ given by (4.61), one has

$$\begin{aligned}\hat{\xi}_i\left(t+\left(\lfloor\bar{m}_i\rfloor+1\right)h\right)=&\hat{x}_i\left(t+\left(\lfloor\bar{m}_i\rfloor+1\right)h-\bar{\tau}_i\right)\\=&\hat{x}_i\left(t+\left(\lfloor\bar{m}_i\rfloor+1\right)h-\bar{m}_ih\right)\\=&\hat{x}_i\left(t+\left(1+\lfloor\bar{m}_i\rfloor-\bar{m}_i\right)h\right)\end{aligned}\tag{4.83}$$

Combining (4.82) and (4.83), $\hat{x}_i(t)$ can be derived as follows

$$\hat{x}_i(t)=\hat{\xi}_i^{(\lfloor\bar{m}_i\rfloor+1)}\left(t+\left(1+\lfloor\bar{m}_i\rfloor-\bar{m}_i\right)h\right),i=1,\ldots,n-1.\tag{4.84}$$

Recall that the above equality is valid only if $\bar{m}_i$ is not an integer.

4.7 Illustrative examples

To this point, a new approach has been presented to deal with a class of uniformly observable systems with delays in the output and in the states. To validate the performance of this observer and the presented methodology, two examples are presented. The first set of simulations deals with an academic system, where different types of configurations and the presence of delays are exploited. The second set of simulations is the simple pendulum.

4.7.1 Academic example

Consider the following third order nonlinear system:

$$\mathcal{SYS}:\left\{\begin{array}{rl}\begin{bmatrix}\dot{x}_1(t)\\\dot{x}_2(t)\\\dot{x}_3(t)\end{bmatrix}&=\begin{bmatrix}x_2(t)-0.5x_1(t)\\x_3(t-\tau_3)+u(t)\sin(x_1(t)x_2(t))\\-0.5x_3(t)+0.5\tanh(x_2(t)x_3(t))-x_1(t-\tau_s)u(t)+\varepsilon(t)\end{bmatrix}\\y_a(t)&=x_1(t-\tau_1)\end{array}\right.\tag{4.85}$$

where $x(t)=[x_1(t)\ x_2(t)\ x_3(t)]^T\in\mathbb{R}^n$ represents the state vector with $n=3$, $u(t)\in\mathbb{R}$ represents the input system and $y_a(t)\in\mathbb{R}$ denotes the available delayed output and its the delayed version of the non-available continuous output $y(t)$. The function $\varepsilon(t)$ is treated as unknown and represents the uncertainties. Notice that the system (4.85) presents different time-delays in its structure; $\tau_1>0$ represents the time-delay in the output, $\tau_3>0$ represents the time-delay in the third state and τ_s represents the time-delay in the state. Note that the second state has no time-delay, i.e. $\tau_2=0$.

In order to obtain the delay-actual state vector of the system (4.85), the following change of variables is defined:

$$\xi(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{bmatrix} = \begin{bmatrix} x_1(t-\bar{\tau}_1) \\ x_2(t-\bar{\tau}_2) \\ x_3(t-\bar{\tau}_3) \end{bmatrix} \tag{4.86}$$

where

$$\begin{aligned} \bar{\tau}_1 &= \textstyle\sum_{i=1}^{1} \tau_i = \tau_1 \\ \bar{\tau}_2 &= \textstyle\sum_{i=1}^{2} \tau_i = \tau_1 + \tau_2 = \tau_1 \\ \bar{\tau}_3 &= \textstyle\sum_{i=1}^{3} \tau_i = \tau_1 + \tau_2 + \tau_3 = \tau_1 + \tau_3 \end{aligned} \tag{4.87}$$

Bearing in mind the above and considering the change of variables (4.86), the system (4.85) can be put in the following form:

$$\left\{ \begin{aligned} \begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \\ \dot{\xi}_3(t) \end{bmatrix} &= \begin{bmatrix} \xi_2(t) - 0.5\xi_1(t) \\ \xi_3(t) + u_{\bar{\tau}_2}(t)\sin(\xi_1(t)\xi_2(t)) \\ -0.5\xi_3(t) + 0.5\tanh(\xi_2(t-\tau_3)\xi_3(t)) - \xi_1(t-\tau_s-\tau_3)u_{\tau_1+\tau_3}(t) + \varepsilon_{\bar{\tau}_3}(t) \end{bmatrix} \\ y_a(t) &= \xi_1(t) \end{aligned} \right. \tag{4.88}$$

with $u_{\bar{\tau}_2}(t) = u(t-\bar{\tau}_2) = u(t-\tau_1)$, $u_{\bar{\tau}_3}(t) = u(t-\bar{\tau}_3) = u(t-\tau_1-\tau_3)$ and $\varepsilon_{\bar{\tau}_3}(t) = \varepsilon(t-\bar{\tau}_3) = \varepsilon(t-\tau_1-\tau_3)$.

Now, the system (4.88) can be expressed as follows:

$$\dot{\xi}(t) = A_3\xi(t) + \varphi(u_{\bar{\tau}_2}(t), u_{\bar{\tau}_3}(t), \xi(t), \xi_\tau(t)) + B_3\varepsilon_{\bar{\tau}_3}(t) \tag{4.89}$$

where

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\varphi(u_{\bar{\tau}_2}(t), u_{\bar{\tau}_3}(t), \xi(t), \xi_\tau(t)) = \begin{pmatrix} -0.5\xi_1(t) \\ u_{\bar{\tau}_2}(t)\sin(\xi_1(t)\xi_2(t)) \\ -0.5\xi_3(t) + 0.5\tanh(\xi_2(t-\tau_3)\xi_3(t)) - \xi_1(t-\tau_s-\tau_3)u_{\bar{\tau}_3}(t) \end{pmatrix} \tag{4.90}$$

Note that the $\varphi(u_{\bar{\tau}_2}(t), u_{\bar{\tau}_3}(t), \xi(t), \xi_\tau(t))$ depends on the actual state vector $\xi(t)$ and its delayed version $\xi_\tau(t)$ defined as follows:

$$\xi_\tau(t) = \begin{bmatrix} \xi_1(t-\tau_s-\tau_3) \\ \xi_2(t-\tau_3) \end{bmatrix}. \tag{4.91}$$

Now, the system (4.89) can be written in the following compact form, which is used for the predictor design:

$$\dot{\xi}^{(j)}(t) = A_3\xi^{(j)}(t) + \varphi(u^{(j)}_{\bar{\tau}_2}(t), u^{(j)}_{\bar{\tau}_3}(t), \xi^{(j)}(t), \xi^{(j-1)}_\tau(t+h)) + B_3\varepsilon_{\bar{\tau}_3}(t) \tag{4.92}$$

where A_3 and B_3 are given in (4.90) and the nonlinear vector function is:

$$\varphi(u^{(j)}_{\bar{\tau}_2}(t), u^{(j)}_{\bar{\tau}_3}(t), \xi^{(j)}(t), \xi^{(j-1)}_\tau(t+h)) =$$
$$\begin{pmatrix} -0.5\xi^{(j)}_1(t) \\ u^{(j)}_{\bar{\tau}_2}(t)\sin(\xi^{(j)}_1(t)\xi^{(j)}_2(t)) \\ -0.5\xi^{(j)}_3(t) + 0.5\tanh(\xi^{(j-1)}_2(t-\tau_3+h)\xi^{(j)}_3(t)) - \xi^{(j-1)}_1(t-\tau_s-\tau_3+h)u^{(j)}_{\bar{\tau}_3}(t) \end{pmatrix} \tag{4.93}$$

Bearing in mind the above, one proceeds to design a cascade observer of the form (4.73) in order to estimate $\xi(t+\bar{\tau}_3) = \xi(t+\tau_1+\tau_3)$.

Notice that all the following simulations that shall be given were obtained without proceeding to the saturation of the nonlinearities and the state of the system is bounded for the considered initial conditions.

The simulations of the system (4.85) are carried out with the following input and initial condition values:

$$u(t) = 0.1\sin(0.1\pi t),$$
$$x(0) = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T,$$

whereas the cascade observer was initialized as follows:

$$\xi^{(j)}(s) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, \text{ for } j = 0, \dots, m, \ s \in \left[-\bar{\tau}_n - j\frac{\bar{\tau}_n}{m}, 0\right].$$

In order to demonstrate the effectiveness of the observer, the unknown uncertain function ε is defined as follows

$$\varepsilon(t) = 0.5\sin(t). \tag{4.94}$$

For the proposed observer, the value of the adjustment parameter θ is set to 10, the value of $K = \begin{bmatrix} 3 & 3 & 1 \end{bmatrix}^T$ is kept constant for all simulations and Δ_θ is defined as follows:

$$\Delta_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\theta} & 0 \\ 0 & 0 & \frac{1}{\theta^2} \end{bmatrix}, \tag{4.95}$$

and the Hurwitz matrix $\bar{A}$ is chosen as follows

$$\bar{A} = A_3 - \lambda I_3 \tag{4.96}$$

where λ is a positive value and is set to 10.

Three simulations are presented to highlight possible cases for state recovery and different scenarios. In the first and second simulation, the following time-delay values are considered: $\tau_1 = 0.4$ sec, $\tau_3 = 0.6$ sec, and $\tau_s = 0.1$ sec. Notice that the output delay τ_1 and the delay in the third state τ_3 are **commensurable**. In this way, it is necessary to define a prediction horizon $h = \frac{\bar{\tau}_n}{m}$ that will be able to reconstruct the three states. For this, one obtains the values of each $\bar{\tau}_i, i = \{1,2,3\}$. Thus, according to (4.87) , one gets:

$$\begin{aligned} \bar{\tau}_1 &= 0.4 \text{ sec} \\ \bar{\tau}_2 &= 0.4 \text{ sec} \\ \bar{\tau}_3 &= 1.0 \text{ sec} \end{aligned} \tag{4.97}$$

According to (4.72), the prediction horizon h must be defined as follows:

$$\begin{aligned} \frac{\bar{\tau}_n}{m} &\leq \min\{\ \tau_s,\ \ \bar{\tau}_1,\ \ \bar{\tau}_2,\ \ \bar{\tau}_3\ \} \\ h &\leq \min\{\ 0.1,\ \ 0.4,\ \ 0.4,\ \ 1.0\ \} \text{ sec} \\ h &\leq 0.1 \text{ sec} \end{aligned} \tag{4.98}$$

Thus, to satisfy the above, $h = 0.05$ sec is proposed. Now, the value of $m_i, i = \{1,2,3\}$ must be obtained. Taking into account the equation (4.74), one gets:

$$\begin{aligned} m_1 &= \frac{\tau_1}{h} = \frac{0.4}{0.05} = 8, \\ m_2 &= \frac{\tau_2}{h} = \frac{0}{0.05} = 0, \\ m_3 &= \frac{\tau_3}{h} = \frac{0.6}{0.05} = 12. \end{aligned}$$

This means that $m_1 = 8$ predictors are needed to deal with the output delay τ_1, $m_2 = 0$ predictors since $\tau_2 = 0$ and $m_3 = 12$ predictors are needed for the last delay in the third state τ_3.

Notice that since the observer is designed for the system (4.92), the number of predictors (m) is chosen so that the cascade can be dealt with $\bar{\tau}_3$. According to equation (4.76), one obtains $\bar{m}_3$, that is, the number of predictors in the cascade as follows:

$$\begin{aligned}\bar{m}_3 &= \sum_{j=1}^{3} m_j \\ &= m_1 + m_2 + m_3 \\ &= 8 + 0 + 12 \\ \bar{m}_3 &= 20.\end{aligned} \tag{4.99}$$

Now, according to the definition of the ξ_i's $i = 1, \ldots, n$ given by (4.86) and using (4.97), the original state components $x_i(t)$'s can be recovered from ξ_i's as follows

$$\begin{aligned}\hat{x}_1(t) &= \hat{\xi}_1\left(t + \bar{\tau}_1\right) = \hat{\xi}_1(t + \bar{m}_1 h) = \hat{\xi}_1^{\bar{m}_1}(t), \\ \hat{x}_2(t) &= \hat{\xi}_2\left(t + \bar{\tau}_2\right) = \hat{\xi}_2(t + \bar{m}_2 h) = \hat{\xi}_2^{\bar{m}_2}(t), \\ \hat{x}_3(t) &= \hat{\xi}_3\left(t + \bar{\tau}_3\right) = \hat{\xi}_3(t + \bar{m}_3 h) = \hat{\xi}_3^{\bar{m}_3}(t),\end{aligned}$$

with $\bar{m}_1 = 8$, $\bar{m}_2 = 8$ and $\bar{m}_3 = 20$.

In the first simulation, one considers that there is no presence of uncertainties, i.e. $\varepsilon(t) = 0$ to demonstrate that in an ideal case the states recovered by the observer will be the same as the original system. The results of the first set of simulations are shown in Figs.4.1-4.3. Figures 4.1 and 4.2 show the estimation of the first and second states and their errors, respectively. These two states are reconstructed in the $j = 8$ predictor. Since the original system does not have the presence of τ_2; therefore, $\bar{\tau}_1 = \bar{\tau}_2$. The last state $x_3(t)$ and its error are shown in Figure 4.3. It is obtained in the third element of the last predictor of the cascade, i.e $j = 20$. It is important to note that the errors converge to zero, since in this simulation, the presence of uncertainties in the system is not considered.

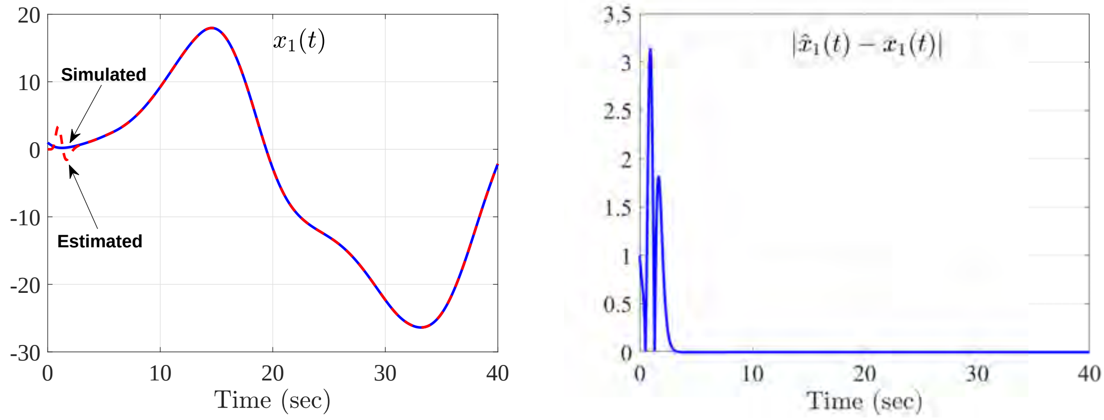

Figure 4.1: Estimation of $x_1(t)$ with the predictor $j = 8$ and its error

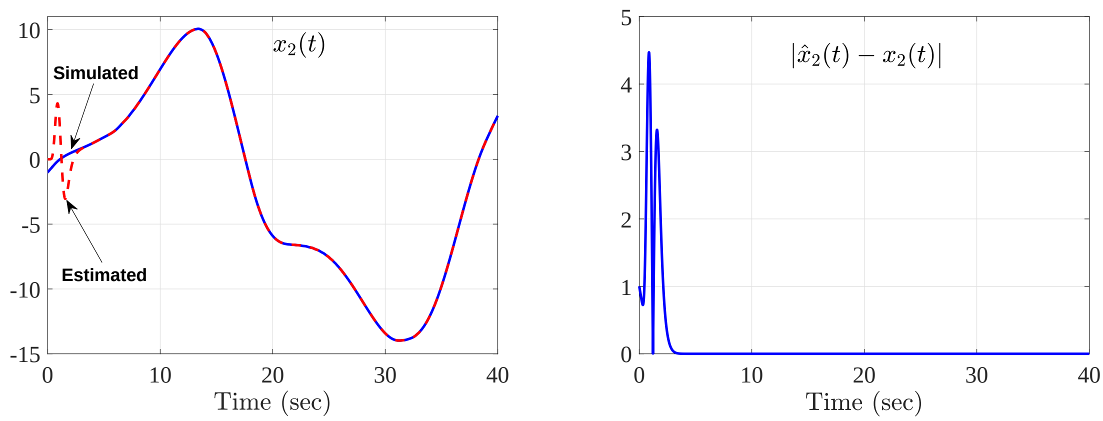

Figure 4.2: Estimation of $x_2(t)$ with the predictor $j = 8$ and its error

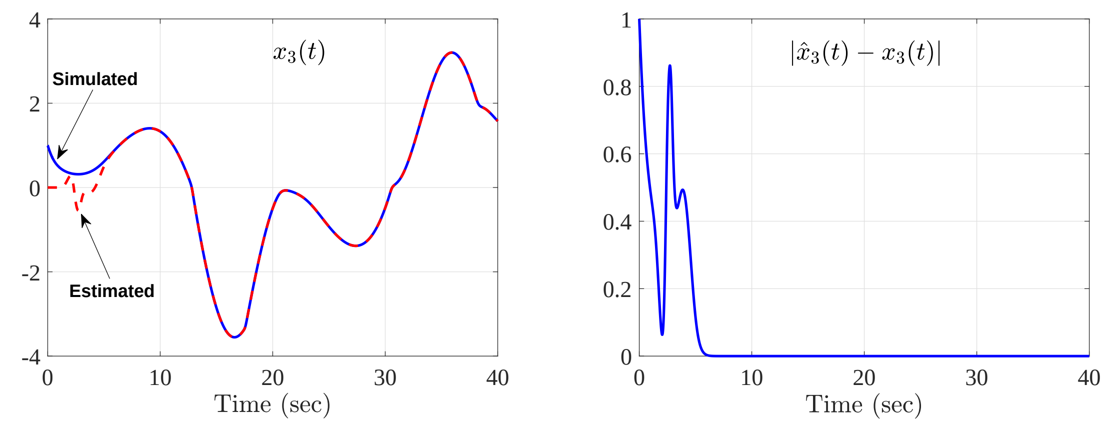

Figure 4.3: Estimation of $x_3(t)$ with the predictor $j = 20$ and its error

For the second and third simulation, one considers that there is uncertainties in the system, i.e. $\varepsilon(t) \neq 0$ and is shown in (4.94). The results of the second set of simulations are shown in Figs.4.4-4.6. The actual estimate of the first state $x_1(t)$ and the second state $x_2(t)$ are shown in Fig.(4.4)-(4.5), respectively. The estimated states are recovered via the first and second elements of the predictor number $j = 8$, i.e., $\hat{x}_1(t) = \hat{\xi}_1^{(8)}(t)$ and $\hat{x}_2(t) = \hat{\xi}_2^{(8)}(t)$. The reason for this is that the original system does not have the presence of τ_2; therefore, $\bar{\tau}_1 = \bar{\tau}_2$, so the number of predictors needed to reconstruct $x_2(t)$ is the same as for $x_1(t)$. In the case that there is a delay in the second state, i.e. $\tau_2 \neq 0$, other predictors would have to be added to reconstruct the original state. The estimation of the third state $x_3(t)$ by means of the third component of the last predictor $j = 20$, i.e. $\hat{x}_3(t) = \hat{\xi}_3^{(20)}(t)$, is shown in Figure 4.6.

The comparison of the first set of simulations with the second set shows that when uncertainties exist in the system, there are differences between the original and estimated states. In the case that no uncertainties exist, the trajectories of the observer should be the same as the original system according to Theorem 4.5.1, as one can see in Figures 4.1-4.3. This simulation shows the ability of the observer to achieve the reconstruction of the original state vector even in the presence of delayed outputs, states, and uncertainties.

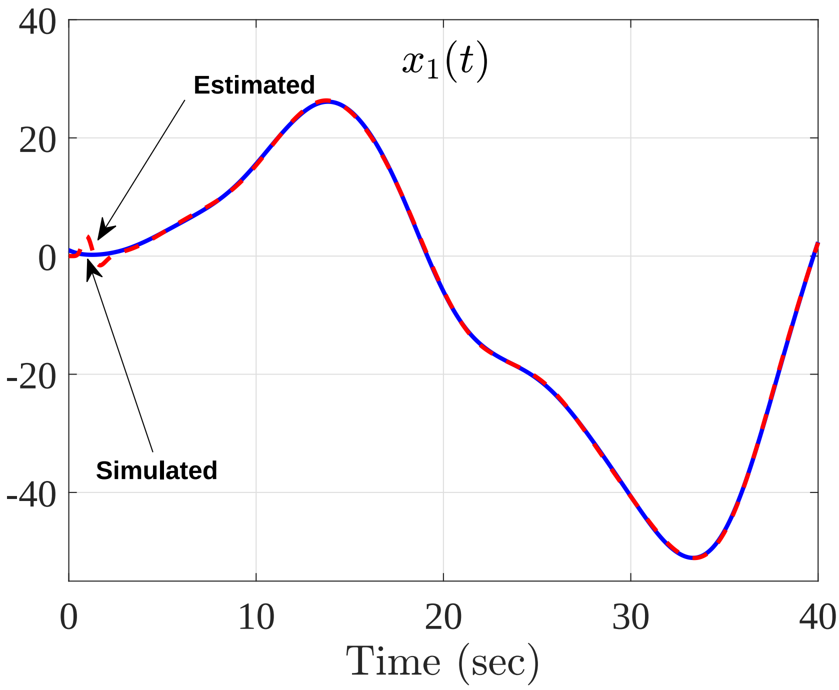

Figure 4.4: Estimation of the state actual state $x_1(t)$ with the predictor number $j = 8$.

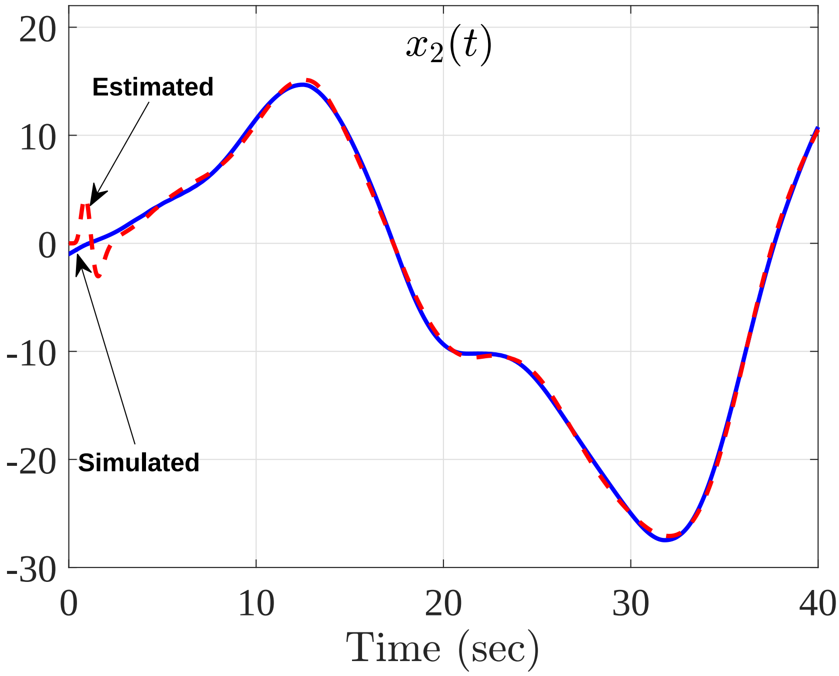

Figure 4.5: Estimation of the state actual state $x_2(t)$ with the predictor number $j = 8$.

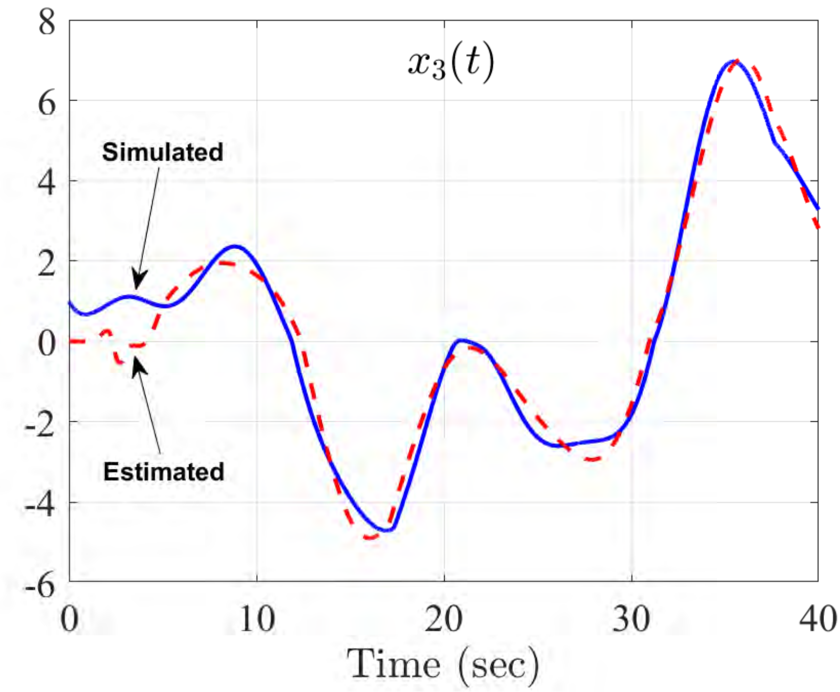

Figure 4.6: Estimation of the state actual state $x_3(t)$ with the predictor number $j = 20$.

The third set of simulation is to demonstrate the second case of the recovery of the state vector. For this purpose, the following values are considered for the delay at the output $\tau_1 = \sqrt{5}$ sec, the third state $\tau_3 = \pi$ sec and the state delay $\tau_s = 1$ sec. Bearing in mind this, it is evident that the system presents **non commensurable** delays, so it is necessary to recover the original states of the system in a different way.

In order to deal with the case non commensurable, the value of the prediction horizon (h) must be first obtained. For this simulation, one considers the number of predictors is $m = 60$, yielding:

$$h = \frac{\bar{\tau}_n}{m} = \frac{\sum_{i=1}^{3} \tau_i}{m} = \frac{\tau_1 + \tau_2 + \tau_3}{m} = \frac{\sqrt{5} + 0 + \pi}{60} \text{ sec} \approx 0.0896 \text{ sec} \tag{4.100}$$

Similarly to commensurable delays, the last state is recovered with the last predictor, i.e.

$$\hat{x}_3(t) = \hat{\xi}_3^{(m)}(t) = \hat{\xi}_3^{(60)}(t).$$

Now, it is necessary to obtain the predictor number to reconstruct the remaining states, i.e. $x_i(t), i = \{1, 2\}$. To this end, one defines each $\bar{m}_i, i = \{1, 2\}$ as follows:

$$\bar{m}_1 = \frac{\bar{\tau}_1}{h} = \frac{\sqrt{5}}{0.0896} = 24.9561,$$

$$\bar{m}_2 = \frac{\bar{\tau}_2}{h} = \frac{\sqrt{5} + 0}{0.0896} = 24.9561.$$

Because $\tau_2 = 0$ the same number of predictors to recover the first state can recover the second state $\bar{m}_1 = \bar{m}_2$, such that

$$\hat{x}_1(t) = \hat{\xi}_1(t + \bar{m}_1 h) = \hat{\xi}_1^{(24.9561)}(t),$$

$$\hat{x}_2(t) = \hat{\xi}_2(t + \bar{m}_2 h) = \hat{\xi}_2^{(24.9561)}(t).$$

As can be seen, the values of $\bar{m}_1 = \bar{m}_2$ are not integers; therefore, the values of $\bar{m}_1 = \bar{m}_2$ do not correspond to a state of the cascade. To deal with this, one proceeds to express it as a subsystem of the cascade as follows

$$\hat{\xi}_1 \left(t + \left(\lfloor m_1 \rfloor + 1\right) h\right) = \hat{\xi}_1^{(24+1)}(t) = \hat{\xi}_1^{(25)}(t),$$

$$\hat{\xi}_2 \left(t + \left(\lfloor m_2 \rfloor + 1\right) h\right) = \hat{\xi}_2^{(24+1)}(t) = \hat{\xi}_2^{(25)}(t).$$

Thus, according to the definition of ξ given by (4.61) and considering (4.83), one can recover the actual state as follows

$$\hat{x}_1(t) = \hat{\xi}_1^{(25)} \left(t - \left(25 - \bar{m}_1\right) h\right),$$

$$\hat{x}_2(t) = \hat{\xi}_2^{(25)} \left(t - \left(25 - \bar{m}_2\right) h\right).$$

In summary, the original state vector is recovered as follows:

$$\begin{aligned} \hat{x}_1(t) &= \hat{\xi}_1^{(25)}\left(t-\left(25-\bar{m}_1\right)h\right), \\ \hat{x}_2(t) &= \hat{\xi}_2^{(25)}\left(t-\left(25-\bar{m}_2\right)h\right), \\ \hat{x}_3(t) &= \hat{\xi}_3^{(60)}(t). \end{aligned} \tag{4.101}$$

The results of the second set of simulations are shown in Figs. 4.7-4.9. The estimations of the first state $x_1(t)$ and the second state $x_2(t)$ are shown in Figs. 4.7-4.8, respectively. It is important to note that these states are recovered as shown in equation (4.101), these states are the delayed version of the $j = 25$ predictor, i.e. $\hat{\xi}_i^{(25)}(t-(25-\bar{m}_i)h), i = \{1, 2\}$. This is because the states are non commensurable, so there is no predictor in the original cascade that is able to reconstruct exactly the delays $\bar{\tau}_1$ and $\bar{\tau}_2$. The estimation of the third state $x_3(t)$ is shown in Fig. 4.9 and is recovered by the last element of the last predictor $j = 60$, i.e. $\hat{x}_3 = \hat{\xi}_3^{(60)}(t)$. As shown in the figures, the estimated states present differences with respect to the original states due to the uncertainties in the system, i.e $\varepsilon(t) \neq 0$. In the case that the uncertainties do not exist, the estimated states will be equal to the original ones.

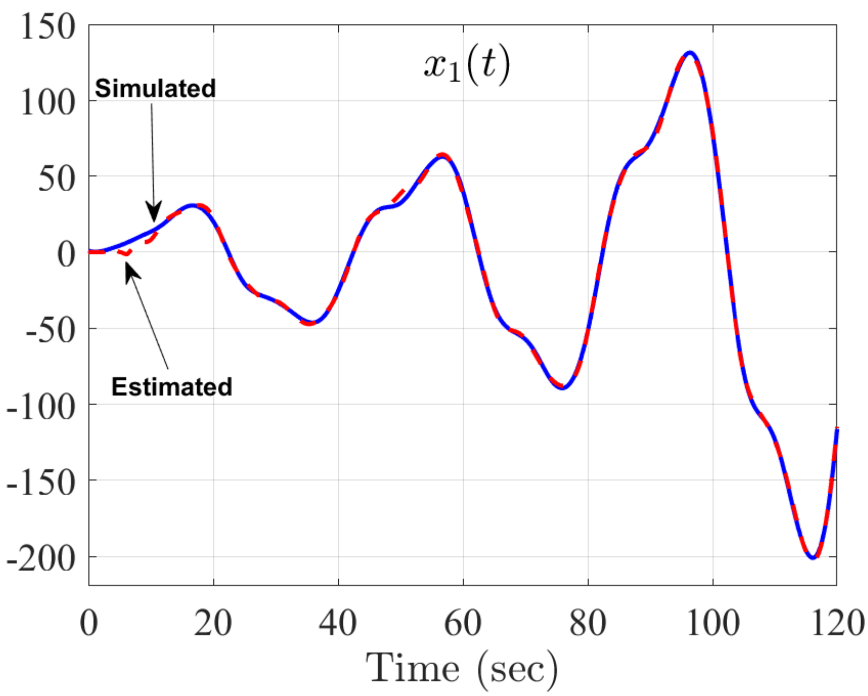

Figure 4.7: Estimation of the state actual state $x_1(t)$ with $m = 60$.

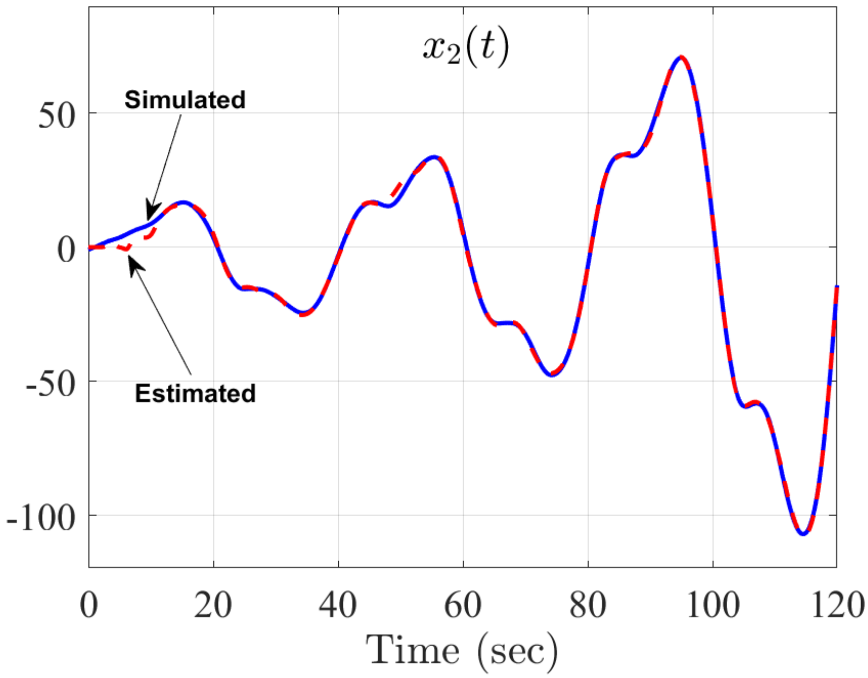

Figure 4.8: Estimation of the state actual state $x_2(t)$ with $m = 60$.

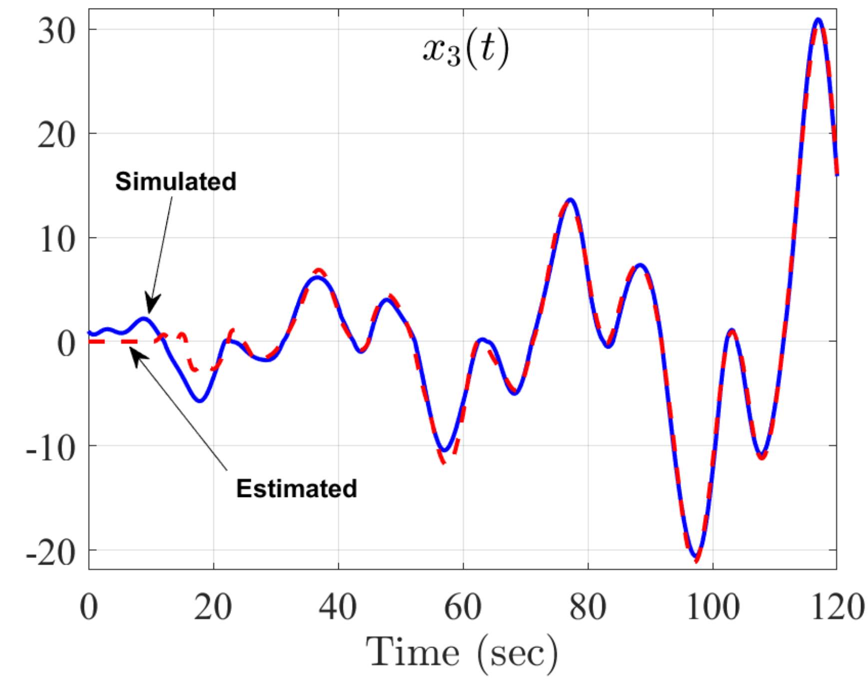

Figure 4.9: Estimation of the state actual state $x_3(t)$ with $m = 60$.

4.7.2 Simple pendulum

Inspired by the proposed in Mohammadi et al. (2021), one considers a time-delayed version of a pendulum system as follows:

$$
\mathcal{SYS}: \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t-\tau_2) \\ \frac{u(t)}{J} - \frac{Mgl}{J}\sin(x_1(t-\tau_s)) - \frac{f_d}{J}x_2(t) + \varepsilon(t) \end{bmatrix} \\ y_{(a)}(t) = \; x_1(t-\tau_1) \end{cases} \tag{4.102}
$$

where $x(t) = [x_1(t),\ x_2(t)]^T \in \mathbb{R}^2$ denotes the actual state vector; $x_1(t)$ represents the angular position of the pendulum and $x_2(t)$ its angular velocity. The delayed available output is denoted by $y_{(a)} \in \mathbb{R}$; the system input is denoted by $u(t) \in \mathbb{R}$ and the unknown function is represented by the function $\varepsilon(t)$. Notice that the system (4.102) presents three time-delay values; $\tau_1 > 0$ represents the output system, $\tau_2 > 0$ represents the delay in the second state and $\tau_s > 0$ represents the delay in the state. The parameters of the system (4.102) are shown in the Table 4.1.

Constant	**Value**	**Unit**
M	1/3	kg
l	2/3	m
g	9.81	$\mathrm{m/sec^2}$
J	0.1975	$\mathrm{kgm^2}$
f_d	0.2	–

Table 4.1: Physical parameters of the pendulum system

In order to obtain the actual state vector of the system (4.102), one defines the following change of variables:

$$
\xi(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t-\bar{\tau}_1) \\ x_2(t-\bar{\tau}_2) \end{bmatrix} \tag{4.103}
$$

where

$$
\begin{aligned} \bar{\tau}_1 &= \textstyle\sum_{i=1}^{1} \tau_i = \tau_1 \\ \bar{\tau}_2 &= \textstyle\sum_{i=1}^{2} \tau_i = \tau_1 + \tau_2 \end{aligned} \tag{4.104}
$$

By considering the change of variables (4.103), the system (4.102) can be rewritten as follows:

$$
\begin{cases} \begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} = \begin{bmatrix} \xi_2(t) \\ \frac{u_{\bar{\tau}_2}(t)}{J} - \frac{Mgl}{J}\sin(\xi_1(t-\tau_s-\tau_2)) - \frac{f_d}{J}\xi_2(t) + \varepsilon_{\bar{\tau}_2}(t) \end{bmatrix} \\ y_a(t) \; = \xi_1(t) \end{cases} \tag{4.105}
$$

with $u_{\bar{\tau}_2}(t) = u(t-\bar{\tau}_2) = u(t-\tau_1-\tau_2)$ and $\varepsilon_{\bar{\tau}_2}(t) = \varepsilon(t-\bar{\tau}_2) = \varepsilon(t-\tau_1-\tau_2)$. Now, the system (4.105) can be expressed as follows:

$$\dot{\xi}(t) = A_2\xi(t) + \varphi(u_{\bar{\tau}_2}(t), \xi(t), \xi_\tau(t)) + B_2\varepsilon_{\bar{\tau}_2}(t) \tag{4.106}$$

where

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\varphi(u_{\bar{\tau}_2}(t), \xi(t), \xi_\tau(t)) = \begin{pmatrix} 0 \\ \frac{u_{\bar{\tau}_2}(t)}{J} - \frac{Mgl}{J}\sin(\xi_1(t-\tau_s-\tau_2)) - \frac{f_d}{J}\xi_2(t) \end{pmatrix} \tag{4.107}$$

Notice that the nonlinear vector function $\varphi(u_{\bar{\tau}_2}(t), u_{\bar{\tau}_3}(t), \xi(t), \xi_\tau(t))$ depends on the actual state vector $\xi(t)$ and the delayed state $\xi_\tau(t) = \xi_1(t-\tau_s-\tau_2)$.

Bearing in mind the above, the system (4.106) can be written in the following compact form, which is used for the predictor design:

$$\dot{\xi}^{(j)}(t) = A_2\xi^{(j)}(t) + \varphi(u^{(j)}_{\bar{\tau}_2}(t), \xi^{(j)}(t), \xi^{(j-1)}_\tau(t+h)) + B_2\varepsilon_{\bar{\tau}_3}(t) \tag{4.108}$$

where A_2 and B_2 are given in (4.107) and the nonlinear vector function is:

$$\varphi(u^{(j)}_{\bar{\tau}_2}(t), \xi^{(j)}(t), \xi^{(j-1)}_\tau(t+h)) = \begin{pmatrix} 0 \\ \frac{u^{(j)}_{\bar{\tau}_2}(t)}{J} - \frac{Mgl}{J}\sin(\xi^{(j-1)}_1(t-\tau_s-\tau_2+h)) - \frac{f_d}{J}\xi^{(j)}_2(t) \end{pmatrix} \tag{4.109}$$

Now, the challenge is to design a cascade observer of the form (4.73) in order to estimate the state vector $\xi(t+\bar{\tau}_2) = \xi(t+\tau_1+\tau_2)$.

The simulations of the system (4.102) are carried out with the following input and initial condition values:

$$u(t) = 0.01\sin(0.1t)$$
$$x(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^T.$$

The cascade observer was initialized as follows:

$$\xi^{(j)}(s) = \begin{bmatrix} 0 & 0 \end{bmatrix}, \text{ for } j = 0, \ldots, m, s \in \left[-\bar{\tau}_n - j\frac{\bar{\tau}_n}{m}, 0\right].$$

The adjustment parameter θ is set to θ and the values of matrix K and Δ_θ are defined as follows:

$$K = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \Delta_\theta = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\theta} \end{bmatrix}, \tag{4.110}$$

and the Hurwitz matrix $\bar{A}$ is chosen as follows:

$$\bar{A} = \bar{A}_2 - \lambda I_2$$

where λ is set to 10.

Two simulations are presented to demonstrate a physical application of the observer. In both cases, the following delays are considered; $\tau_1 = \sqrt{2}$ sec, $\tau_2 = 0.5$ sec and $\tau_s = 0.5$ sec. In this sense, since τ_1 is **non commensurable**, the cascade must be designed for that case.

In order to obtain the value of the predictor horizon (h), one considers the number of the predictor as $m = 30$. Now, obtaining the value of h as follows:

$$h = \frac{\bar{\tau}_n}{m} = \frac{\sum_{i=1}^{2} \tau_i}{m} = \frac{\tau_1 + \tau_2}{m} = \frac{\sqrt{2} + 0.5}{30} \text{ sec} \approx 0.0638 \text{ sec}. \tag{4.111}$$

The last state $x_2(t)$ is recovered with the last predictor, i.e.

$$\hat{x}_2(t) = \hat{\xi}_2^{(30)}(t).$$

Now, it is necessary to obtain the value of $\bar{m}_1$ in order to reconstruct the state $\xi_1(t+\bar{m}_1 h)$, $\bar{m}_1$ is obtained:

$$\bar{m}_1 = \frac{\tau_1}{h} = \frac{\sqrt{2} \text{ sec}}{0.0638 \text{ sec}} = 22.1639 \tag{4.112}$$

As it is evident, $\bar{m}_1$ is not an integrator, therefore, one considers the following:

$$\hat{\xi}_1\left(t + \left(\lfloor m_1 \rfloor + 1\right) h\right) = \hat{\xi}_1^{(22+1)}(t) = \hat{\xi}_1^{(23)}(t).$$

Now, according to the definition of $\xi(t)$ given by (4.61) and considering (4.83), one can recover the actual state as follows

$$\hat{x}_1(t) = \hat{\xi}_1^{(23)}\left(t - (23 - \bar{m}_1)\, h\right).$$

Thus, the original state vector is recovered as follows:

$$\begin{aligned} \hat{x}_1(t) &= \hat{\xi}_1^{(23)}\left(t - (23 - \bar{m}_1)\, h\right), \\ \hat{x}_2(t) &= \hat{\xi}_2^{(30)}(t). \end{aligned} \tag{4.113}$$

In the first simulation, the presence of uncertainty in the system is considered, ε is defined as follows:

$$\varepsilon(t) = 0.5 \sin(10t). \tag{4.114}$$

The results of the first simulation are shown in Figs. 4.10-4.11. The estimations of the angular position $x_1(t)$ and angular velocity $x_2(t)$ are shown in Figs. 4.10-4.11, respectively. Since the system presents a non commensurable delay in τ_1, the angular position is recovered in a delayed version of the predictor $j = 23$, i.e. $\hat{\xi}_1^{(23)}(t - (23 - \bar{m}_1)h)$ as shown in Fig. 4.10. The estimation of the angular velocity $x_2(t)$ is shown in Fig. 4.11 and is recovered by the last element of the last predictor $j = 30$, i.e. $\hat{x}_2 = \hat{\xi}_2^{(30)}(t)$. Due to the uncertainties, the estimated states present differences with respect to the original states due to the uncertainties in the system; this is clearly shown in Fig. 4.11.

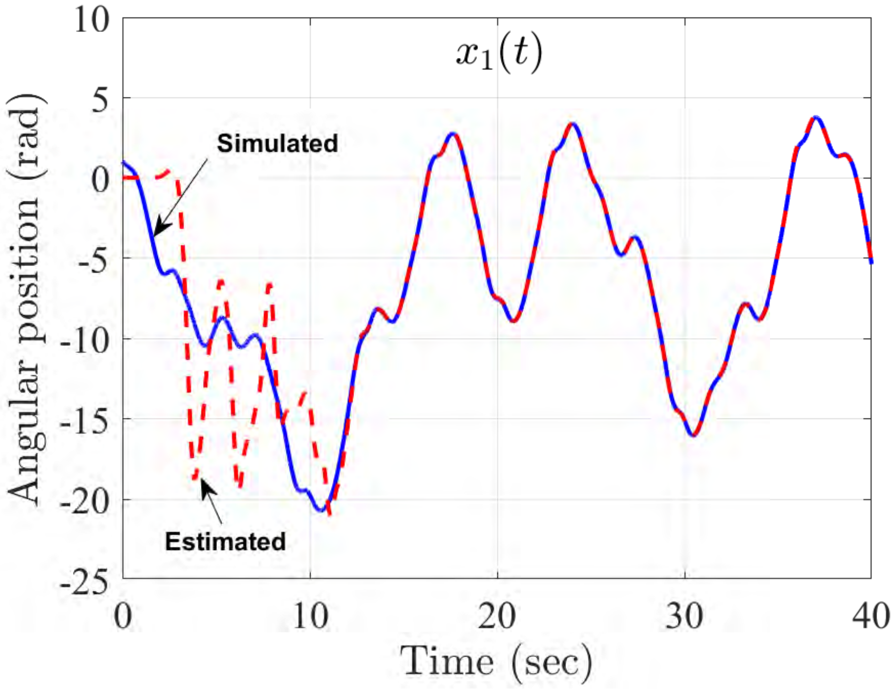

Figure 4.10: Estimation of the state actual angular position $x_1(t)$ with $m = 30$.

The second simulation is focused on demonstrating that in the case of no uncertainties in the system, the observation error will converge to zero. For this, all the above initial conditions are considered in the same way, but the uncertain function is set to zero, i.e. $\varepsilon(t) = 0$. The results of the simulation are shown in Figs. 4.12 and 4.13. The estimations of the angular position $x_1(t)$ and angular velocity $x_2(t)$ are shown in Figs. 4.12 and 4.13, respectively. The states are recovered as in the previous simulation. In order to see better the phenomenon of the presence of uncertainty in the estimation, the norm of the observation error is shown in Figure 1. If the system has uncertainties ($\varepsilon \neq 0$), the norm of the error is bounded to a region. This is in contrast to what is shown when $\varepsilon = 0$ since it converges to zero. It is important to note that in both cases the actual state vector is obtained even in the presence of output and state delays.

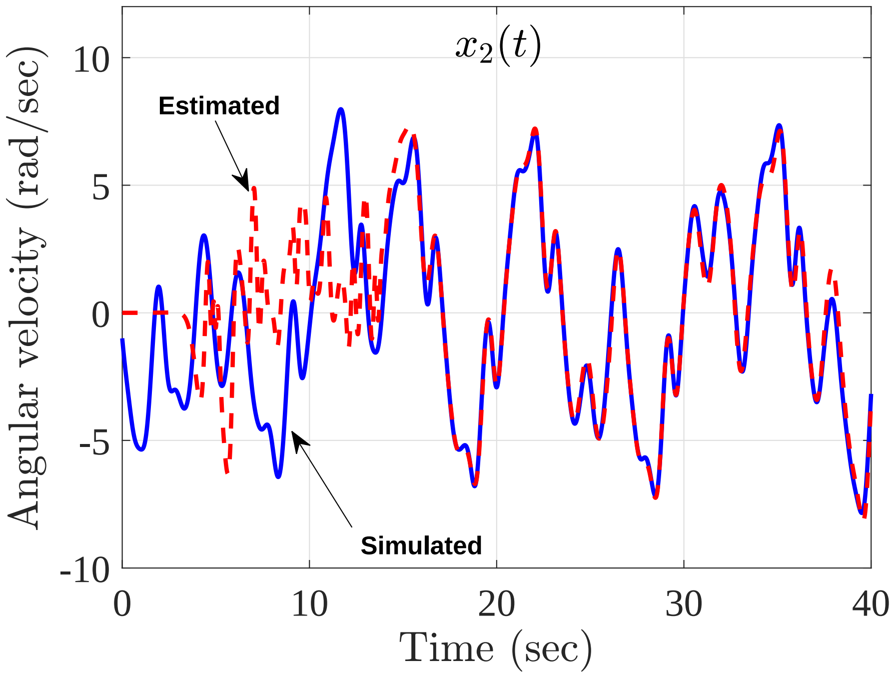

Figure 4.11: Estimation of the state actual angular velocity $x_2(t)$ with $m = 30$.

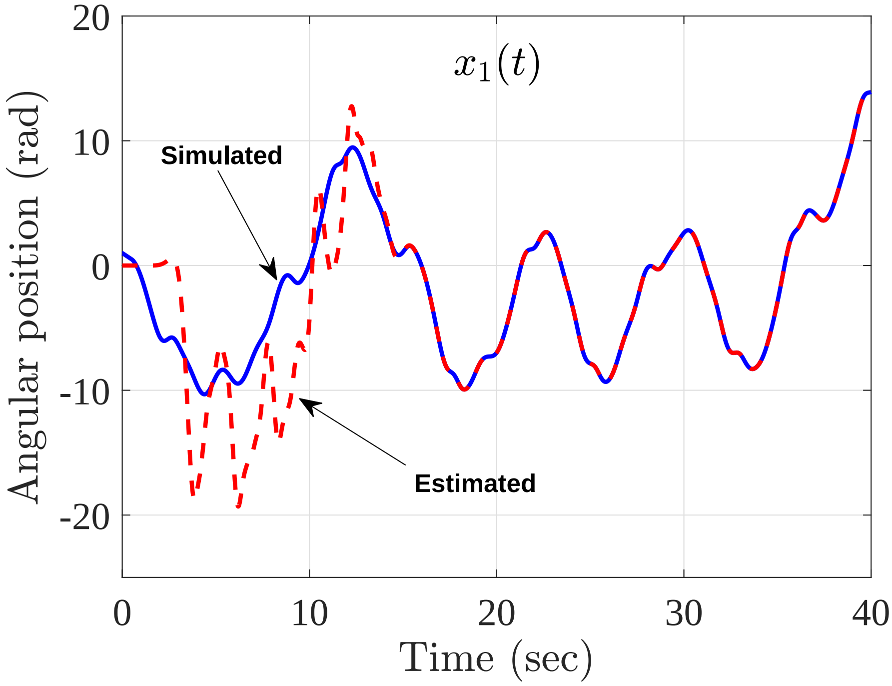

Figure 4.12: Estimation of the state actual angular position $x_1(t)$ with $m = 30$ and without uncertainties.

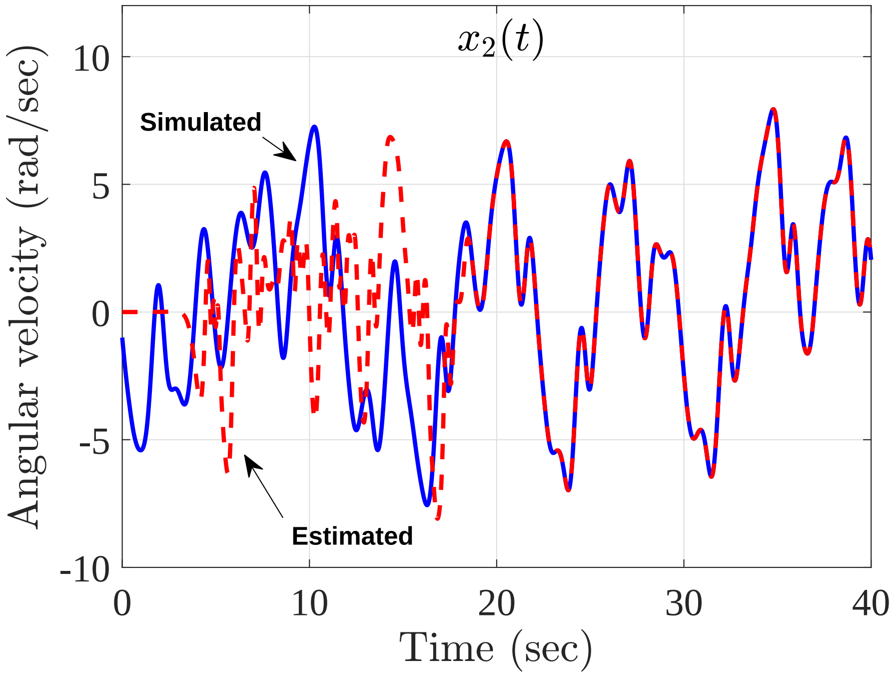

Figure 4.13: Estimation of the state actual angular velocity $x_2(t)$ with $m = 30$ and without uncertainties.

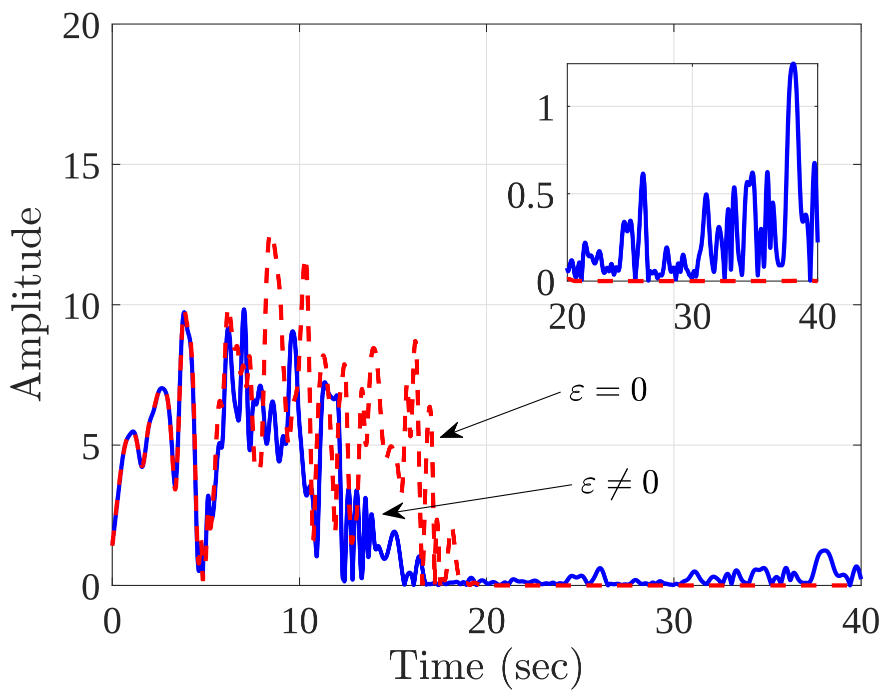

Figure 4.14: Comparison of the norm of the observation error $\|x(t) - \hat{\xi}(t)\|$ with and without uncertainties.

4.8 Conclusion

This chapter presented the design of an observer for an uncertain uniformly observable system in the presence of long delays in output and state. This problem was approached considering a cascade structure, achieving the estimation of the actual state vector of the original system. In this chapter, two design cases focused on the state vector related to the linear part of the dynamic system, whether without delay or with delay. The required methodology to deal with these problems was presented, showing the manner to recover the states corresponding to the original system. It is shown that in case of uncertainties, the estimation error will converge to a bounded region. The effectiveness of the observer is demonstrated through an academic example and the simple pendulum.

Chapter 5

General conclusion

In this thesis work, the development of observers with the high-gain approach for nonlinear systems is presented. In these systems, different practical phenomena are considered, such as multiple sampling, multiple delays, unknown delays, and the presence of delays in different parts of the system.

The first contribution of this work is introduced in Chapter 2. In this chapter, the design of a high-gain observer is developed for a class of nonlinear systems that are non-uniformly observable. A distinguishing aspect of this system lies in the presence of multiple sampling rates and time-varying delays in the measured outputs. This framework closely reflects practical scenarios, where heterogeneity in sensing devices or inconsistencies in instrumentation may naturally lead to asynchronous measurements and variable latencies. The particularity of this class of systems lies in the fact that the part at the end of the state vector can have inputs and states, which relaxes the change of coordinates for the design of the observer. However, it is necessary to satisfy the condition of persistent excitation, as shown in (Hernández-González et al., 2016; Farza et al., 2019). This chapter first addresses the simple case, considering that all outputs have the same sampling intervals and delay functions, to present the necessary considerations for the design and the observer equations. Subsequently, it is extended to the multiple case, which is a more challenging case, so the observer is redesigned to be able to deal with it. It is established that the observation error tends to zero and that the convergence rate depends on the maximum sampling and delay rates present in each available output.

The second contribution is shown in chapter 3. This chapter addresses one of the most interesting and open problems in delay systems, which is the presence of unknown delay in the output signal. In this chapter, the design of an observer for the joint estimation of the delay-free vector and the value of the unknown delay present in the output for a class of uniformly observable nonlinear systems is presented. The observer presented in this chapter is a cascade observer, consisting of a head observer and a number "m" of predictors as proposed in (Farza et al., 2015a; Hernández-González et al., 2016; Germani et al., 2002). However, a redesign in the head observer is proposed. This redesign consists of considering an adaptive observer for the joint estimation of the delayed vector and the value of the unknown delay. The necessary considerations for the design are presented, since, being an adaptive observer, it is necessary to fulfill the condition of persistent excitation for the correct estimation of the unknown delay. It is shown that the observation error and the state error converge exponentially to zero. Thus, the observer in the base achieves the estimation of the delayed state vector and the value of the unknown delay. Subsequently, through the cascade predictors, the delay present in the system is compensated in a delay fraction equal to $\hat{\tau}/m$, achieving with the last predictor of the cascade the estimation of the actual state vector of the system. It is important to emphasize that the compensation is done considering the value of the delay estimated by the head of the observer. Similarly, it is established that the observation error in each predictor converges to zero. Therefore, with this structure, the objective of joint estimation of the actual state vector of the system and unknown delay is achieved.

The third contribution is shown in chapter 4. In this chapter, the design for a class of uniformly observable nonlinear systems in the presence of uncertainties with delayed output and system states is presented. This design mainly addresses a more open and less restrictive problem than other approaches, since in this work, it is considered that the class of nonlinear systems does not necessarily assume that the delays are only present in the vector of nonlinear functions φ but can be present in the linear part. It is important to emphasize that this work deals with the most general case, since the presence of the delays in this part may vary, i.e., they do not necessarily have to be in all the states at the same time, but may be present in some and not in others. The proposed observer is inspired by the cascade observer, redesigning it to deal with the presence of delayed states in the nonlinear function and also with the presence of delays in the linear part. Through a change of coordinates, the system is brought to a triangular form, and then the observer is applied to estimate each actual state of the system. As in the traditional structure, the cascade observer consists of an observer at the head and a number "m" of predictors that compensate in a fraction of time equal to the delay present in the system. However, in this case, the value of the delay of the designed cascade is equal to the maximum delay in the system, considering the delay at the output of the system and each delay present in the linear part of the system. The main difference of this structure is that each actual state is reconstructed in different subsystems within the cascade and not in the last subsystem of the cascade as in the traditional case. Likewise, different cases are presented for the reconstruction of the states, when they are commensurable and non-commensurable, which addresses the cases that may occur. It is shown that the observation error will converge to a bounded region due to the presence of system uncertainties and that only in the case where they do not exist, convergence to zero is achieved.

The performances of the different observers presented and evaluated in different scenarios have been illustrated through the HIV infection model, the model of a robotic arm, the simple pendulum, and academic examples. Highlighting their performance and their application in different kinds of systems.

5.1 Future works

Considering the problems addressed in this thesis, several directions emerge for future research. Some of these are outlined below:

1. Building on the results presented in Chapter 2, future work may focus on observer designs capable of handling long time delays that do not satisfy the inequality conditions established in the corresponding theorems.
2. Based on the developments in Chapter 3, two main research extensions are proposed. The first involves redesigning the observer to account for signals that are not only delayed but also sampled. The second is to extend the approach to a class of non-uniformly observable systems.
3. With respect to Chapter 4, a promising direction would be to adapt the observer design to handle output signals that are sampled rather than continuously available. Additionally, extending the approach to systems with multiple inputs and multiple outputs (MIMO) would significantly enhance its applicability.

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Appendix A

Background on nonlinear systems and observability

Several methods for the control or estimation of linear systems have been generated over time; however, these systems are limited to represent a more complex range of system dynamics. Therefore, the scientific community has focused its efforts on designing methods for nonlinear systems, since these can represent more complex systems such as those found in some areas, such as the aerospace industry, robotics, process control, biomedical engineering, etc.

A nonlinear system is one that does not satisfy the principle of superposition (Khalil and Grizzle, 2002), unlike linear systems. The general representation is as follows:

$$\mathcal{SYS}:\begin{cases}\dot{x}(t)= & f(t,x,u)\\ y(t)= & h(t,x,u)\end{cases} \tag{A.1}$$

where the state $x(t) \in \mathcal{X}$ is an analytic variety of dimension n of class $\mathcal{C}^\infty$, i.e., $\mathcal{X} \subset \mathbb{R}^n$; the input is denoted by $u(t) \in \mathcal{U} \subset \mathbb{R}^m$; the output is denoted by $y(t) \in \mathbb{R}^p$. The vector field $f_u(x) = f(u,x)$ and the function $h = \begin{pmatrix} h_1 \\ \vdots \\ h_q \end{pmatrix}$ are of class $\mathcal{C}^\infty$ on $\mathcal{X}$. The variable t denotes the time. This class of systems can represent a variety of nonlinear phenomena such as:

- **Finite escape time.** A state of a nonlinear system can go to infinity in finite time.
- **Multiple isolated equilibria.** A nonlinear system can have more than one equilibrium point, the state can converge to different equilibrium points depending on its initial condition.

- **Limit cycles.** A nonlinear system can present and maintain sustained oscillations of constant amplitude and frequency, regardless of the initial state.
- **Subharmonic, harmonic or almost-periodic oscillations.** A nonlinear system under periodic excitation may oscillate with frequencies that are sub-multiples or multiples of the frequency of the input signal.
- **Chaos.** A nonlinear system can have chaos as a stable state, despite the nature of the system.
- **Multiple modes of behavior.** Nonlinear systems can exhibit different behaviors depending on whether the system is excited or not. Likewise, it presents changes in response to the amplitude or frequency of the excitation signal.

Considering the general representation shown in A.1, the system can be presented in different ways. Some state-space representations are Besançon (2007):

- Control-affine systems:

$$\mathcal{SYS}: \begin{cases} \dot{x}(t) = & f(x) + g(x)u \\ y(t) = & h(x,u) \end{cases}$$

  In this class of systems, a function $g(x)$ that is affine to the control input u can be separated.

- State-affine systems:

$$\mathcal{SYS}: \begin{cases} \dot{x}(t) = & A(u)x + B(u) \\ y(t) = & Cx(t)(\text{or } C(u)x + D(u)) \end{cases}$$

  In this class of systems, a matrix $A(u)$ is affine to state vector x.

- Linear Time-Varying (LTV) systems:

$$\mathcal{SYS}: \begin{cases} \dot{x}(t) = & A(u)x + B(u)x \\ y(t) = & C(t)x + D(u) \end{cases}$$

  In this class of systems, the matrices A, B, C, D can change over time.

If the functions f or h do not depend on u the system is said to be uncontrollable.

A.1 Observer problem

In practice, the inputs and outputs of the system are usually available. However, in some cases, there are certain variables of interest that are not available. Expressed in variables, one has access to u and y; nevertheless, one wants to know x. Taking this into account, the observation problem can be formulated as follows:

> *Given a system described by A.1, find an estimated state $\hat{x}(t)$ for $x(t)$ from the knowledge of the available outputs $y(\tau)$ and the inputs $u(\tau)$ for $0 \leq \tau \leq t$.*

Based on the aforementioned, it is possible to define what an observer is. An observer is a device or computational algorithm that achieves the estimation of the state vector $x(t)$ of system A.1 through knowledge of the output $y(t)$, the input $u(t)$ and the model of system A.1. A schematic of the system and observer is shown in Figure A.1.

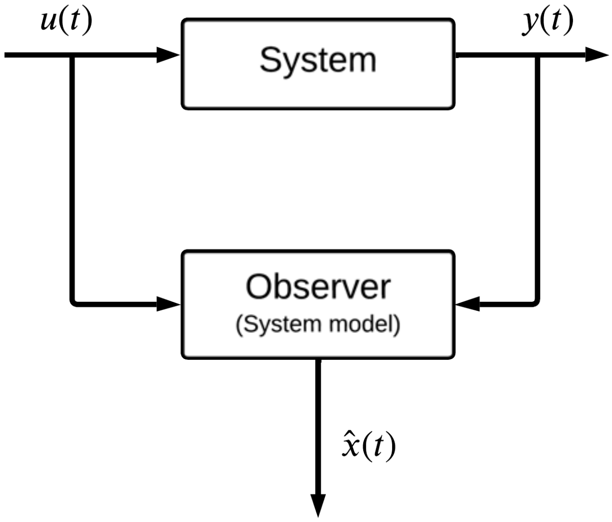

Figure A.1: Observer based state estimation $x(t)$ scheme

A.2 Observability

In order for the observer to achieve his purpose of estimation, some conditions of observability must be ensured. The observability of a system is related to the ability to recover $x(t)$ from the only knowledge of $u(t)$ and $y(t)$ up to time t. This is possible only if the output $y(t)$ has the information of the full state vector.

For a system to be observable in this it must be possible to distinguish between several initial states from the output, the systems must not be **indistinguishable**, this is defined below (Besançon, 2007; Bernard, 2019).

Definition A.2.1 ***Indistinguishability***
A pair of initial conditions $(x_0, x'_0) \in \mathbb{R}^n x \in \mathbb{R}^n$ *is indistinguishable for a system (A.1) if:*

$$\forall u \in \mathcal{U},\ \forall t \geq 0,\ h(\mathcal{X}_u(t, x_0)) = h(\mathcal{X}_u(t, x'_0)) \tag{A.2}$$

A state x *is indistinguishable from* x_0 *if the pair* (x, x_0) *is indistinguishable.*

It is also necessary to define the concept of *detectability* since there are some systems that do not comply with the *indistinguishability* property but are *detectable.*

Definition A.2.2 ***Detectability***
The system (A.1) is detectable for any $u \in \mathcal{U}$*; i.e., for any* u *in* $\mathcal{U}$ *and for any* (x_a, x_b) *in* $\mathcal{X}_0 \times \mathcal{X}_0$ *such that* $\sigma^+(x_a, u) = \sigma^+(x_b, u) = +\infty$ *and*

$$y_{x_a,u}(t) = y_{x_b,u}(t) \quad \forall t \geq 0 \tag{A.3}$$

then, one has:

$$\lim_{t\to\infty} |X(x_a; t; u) - X(x_b; t; u)| = 0. \tag{A.4}$$

This property indicates that even if two initial conditions are indistinguishable with the output, the corresponding solutions of the system are asymptotically close; therefore, a good estimate can be obtained no matter which condition is chosen.

Definition A.2.3 ***Observability [resp. at*** x_0***]***
A system A.1 is observable (with respect to x_0*) if it does not admit any indistinguishable pair (with respect to any indistinguishable state of* x_0*).*

Example. Let us consider the following system:

$$\mathcal{SYS} : \begin{cases} \dot{x}(t) = u(t) \\ y(t) = \sin(x(t)) \end{cases}$$

and taking into account the following initial states: $x_0 = 0, x'_0 = x_0 + 2\pi$, the output is evaluated with each initial condition:

$$\begin{aligned} \sin(x_0) &= \sin(x'_0) \\ \sin(0) &= \sin(2\pi) \\ 0 &= 0. \end{aligned}$$

It is clear that the output function y does not help to distinguish between the two states x_0 and x'_0, therefore the system is not observable. However, it is clear that it is possible to distinguish the states when $]-\frac{\pi}{2}, \frac{\pi}{2}[$. This gives rise to the following definition.

Definition A.2.4 ***Weak observability [resp. at*** x_0***]***
A system (A.1) is weakly observable (with respect to x_0*) if there exists a neighborhood* U *for any* x *(with respect to* x_0*) such that there exists no indistinguishable state of* x *(with respect to* x_0*) in* U*.*

In order to prevent loss of observability in certain cases, the following is defined.

Definition A.2.5 ***Local weak observability [resp. at*** x_0***]***
A system (A.1) is locally observable (with respect to x_0*) if there exists a neighborhood* U *for any* x *(with respect to* x_0*) such that for any neighborhood* V *of* x *(with respect to* x_0*) contained in* U*, there exists no indistinguishable state of* x *(with respect to* x_0*) in* V *when considering time intervals for which the trajectories remain in* V*.*

This means that each state can be distinguished from its neighbors without going too far. This condition is based on the notion of an observation space.

Definition A.2.6 ***Observation space***
The observation space for the system (A.1) is defined as the smallest vector space (denoted by $\mathcal{O}$*) of* $\mathcal{C}^\infty$ *functions containing the components of* h *and is closed under Lie derivatives along* $f_u := f(.,u)$ *for any constant* u *(such that for any* $\varphi \in \mathcal{O}(h), L_{f_u}\varphi \in \mathcal{O}(h)$*, where* $L_{f_u}\varphi(x) = \frac{\partial \varphi}{\partial x} f(x,u)$*).*

Definition A.2.7 ***Observability rank condition [resp. at*** x_0***]***
A system (A.1) is said to satisfy the rank condition of observability (with respect to x_0*) if:*

$$\forall x, \quad dim\ d\mathcal{O}(h)|_x = n \quad [resp.\ dim\ d\mathcal{O}(h)|_{x_0} = n] \tag{A.5}$$

where $d\mathcal{O}(h)|_x$ *is the set of* $d\varphi(x)$ *with* $\varphi \in \mathcal{O}(h)$*.*

Example. Let us consider the following system:

$$\mathcal{SYS} : \begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t), \quad \text{with } x \in \mathbb{R}^n \end{cases} \tag{A.6}$$

For the above system, the range of conditions is equivalent to weak local observability (which itself is equivalent to observability) and is characterized by the so-called *Kalman rank condition:*

Theorem A.2.1 *For a system of the form A.6*

- *The observability rank condition is equivalent to rank $\mathcal{O}_m = n$ with $\mathcal{O}_m$ the so called* ***observability matrix*** *defined by*

$$\mathcal{O}_m = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

- *The observability rank condition is equivalent to the observability of the system.*

The first point results from the direct calculation of the derivatives of lie since here the kth Lie derivation $L_f^k h(x) = CA^k x$, while the second one results from the definition of observability.

Notice that if system A.6 satisfies the above observability rank condition, the pair $\{A, C\}$ is usually called *observable*.

A.2.1 Analytic condition for Observability

There are some systems for which the above definitions are not sufficient to ensure their observability, and new ones must be defined.

Example. Let us define the following system:

$$\mathcal{SYS} : \begin{cases} x(t) = & \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} x(t) \\ \\ y(t) = & \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases} \tag{A.7}$$

This system is observable for any constant input $u \neq 0$, but not for $u = 0$.

For this class of systems, some more definitions are needed.

Definition A.2.8 ***Universal inputs [resp. on*** $[0, t]$***]***
An input u is an universal input (resp. on $[0, t]$) for a system (A.1) if $\forall x_0 \neq x_0'$, $\exists \tau \geq 0$ (resp. $\exists \tau \in [0, t]$) s.t. $h(\chi_u(\tau, x_0)) \neq h(\chi_u(\tau, x_0'))$. An input u is a singular input if it is not universal.

Definition A.2.9 ***Uniformly observable systems [resp. locally]***
A system is uniformly observable (UO) if every input is universal (resp. on $[0, t]$).

An example of this type of system is as follows:

$$\mathcal{SYS}:\begin{cases} x(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ & & \ddots & \ddots & \\ & & & & 0 \\ \vdots & & & & 1 \\ 0 & \dots & & & 0 \end{bmatrix} x(t) + \begin{pmatrix} \varphi_1(x_1) \\ \varphi_2(x_1, x_2) \\ \vdots \\ \varphi_{n-1}(x_1, \dots, x_{n-1}) \\ \varphi_n(x_1, \dots, x_n) \end{pmatrix} \\ y(t) = \ x_1(t); \text{ with } x = (x_1, \dots, x_n)^T \end{cases} \tag{A.8}$$

For non-uniformly observable systems such as the one shown in A.7, it is necessary to ensure that the input u is persistent over time. Therefore, we define the following:

Proposition A.2.1 *An input u is a universal input on $[0,t]$ for system A.1 if and only if*

$$\int_0^t ||h(\mathcal{X}_u(\tau, x_0)) - h(\mathcal{X}_u(\tau, x_0'))||^2 d\tau > 0, \ \forall x_0 \neq x_0' \tag{A.9}$$

Bearing in mind the above, one can define persistency as follows:

Definition A.2.10 ***Persistent inputs***
An input u is a persistent input to the system (A.1) if

$$\int_{t-T}^t ||h(\mathcal{X}_u(\tau, x_{t-T})) - h(\mathcal{X}_u(\tau, x_{t-T}'))||^2 d\tau > 0, \forall x_{t-T} \neq x_{t-T}' \tag{A.10}$$

In some observers where the gain terms are time-varying, i.e. $K(t)$, it may grow. To avoid this, one defines:

Definition A.2.11 ***Regularly persistent inputs***
An input u is a regularly persistent input to the system (A.1) if

$$\exists t_0, T : \forall x_{t-T}, x_{t-T}', \forall t \geq t_0, \int_{t-T}^t \|h(\mathcal{X}_u(\tau, x_{t-T})) - h(\mathcal{X}_u(\tau, x_{t-T}'))\|^2 d\tau > \beta(\|x_{t-T}, x_{t-T}')\|) \tag{A.11}$$

for some class $\mathcal{K}$ function β.

For state-affine systems, we have the following proposition.

Proposition A.2.2 *For state affine systems, regularly persistent inputs are inputs u such that:*

$$\exists t_0, T, \alpha : \int_{t-T}^{T} \Phi_u^T(s, t-T) C^T C \Phi_u(s, t-T) ds \geq \alpha I, \forall t \geq t_0, \tag{A.12}$$

with $\Phi_u(s,t)$ the transition matrix classically defined by:

$$\frac{d\Phi_u(s,t)}{ds} = A(u(s))\Phi_u(s,t),$$
$$\Phi_u(t,t) = I$$

The quantity of the left-hand side of (A.12) corresponds to what is known as the *gramian of observability*, classically defined for LTV (Linear Time-Variant) systems, for any $t_1 < t_2 \in \mathbb{R}$, as:

$$\Gamma(t_1, t_2) = \int_{t_1}^{t_2} \Phi^T(s, t_1) C^T(s) C(s) \Phi(s, t_1) ds. \tag{A.13}$$

It is important to emphasize that this condition must be satisfied in order to be able to reconstruct the state vector of the systems. If this condition is not satisfied, there is a risk that the observability of the system will be lost. Similarly, the input must be able to excite the system sufficiently to achieve the correct estimation of the state vector. In the case that a system does not satisfy this property *a priori*, it can be made to satisfy it through the system inputs *a posteriori*.

Appendix B

Technical lemmas

B.1 Lemmas of the Chapter 2

Lemma B.1.1 *Ramírez-Rasgado et al. (2021) Consider a differentiable function* $\upsilon : t \in [t_0 - \delta, +\infty[\mapsto \upsilon(t) \in \mathbb{R}^+$ *with* $t_0, \delta \geq 0$*, that satisfies the following inequality*

$$\forall t \geq t_0,\ \dot{\upsilon}(t) \ \leq\ -a\upsilon(t) + b\int_{t-\delta}^{t} \upsilon(s)\ ds + p(t), \tag{B.1}$$

where $a > 0$*,* $b \geq 0$ *and* $p : \mathbb{R}^+ \to \mathbb{R}^+$ *is an essentially bounded function with* $\|p\|_\infty = Ess \sup\limits_{t \geq t_0} p(t) \leq \delta_p$*. If* $\dfrac{b\delta}{a} < 1$*, then the function* υ *satisfies the following property:*

$$\upsilon(t) \leq (1 + \delta\,(a - \eta(\delta)))\, e^{-\eta(\delta)(t-t_0)} \max_{\nu \in [t_0-\delta, t_0]} \upsilon(\nu) + \frac{\delta_p}{\eta(\delta)}, \tag{B.2}$$

with

$$0 < \eta(\delta) = (a - b\delta)\, e^{-a\delta} = a\left(1 - \frac{b\delta}{a}\right) e^{-a\delta} \leq a. \tag{B.3}$$

B.2 Lemmas of the Chapter 4

Lemma B.2.1 *Farza et al. (2018a) Consider a continuous real function* $x : t \in [-\delta, +\infty[\to x(t) \in [0, +\infty[$ *with* $\delta > 0$*, such that*

$$x(t) \leq \rho e^{-at} + M + \eta \int_{t-\delta}^{t} e^{-a(t-s)} x(s) ds \ \ for\ \ t \geq 0, \tag{B.4}$$

where $\rho, M \geq 0$ *and* $a, \eta > 0$ *are real numbers. If* $\eta\delta < 1$ *then, one has*

$$x(t) \leq \frac{1}{1 - \eta\delta} \left\{ \left(\rho + \eta \int_{-\delta}^{0} x(s) ds \right) e^{-at} + M \right\}. \tag{B.5}$$

B.3 Lemmas of the Chapter 5

The next technical lemma will be used in the proof of Theorem 4.3.1.

Lemma B.3.1 *Consider a differentiable function $v : t \in [-\delta, +\infty[\mapsto v(t) \in \mathbb{R}^+$ with $\delta > 0$, satisfying inequality*

$$\forall t \geq 0, \dot{v}(t) \leq -\theta a v(t) + bu(t-\delta) + p(t), \tag{B.6}$$

where $\delta > 0, a > 0, b \geq 0, \theta \geq 1$ is a design parameter and $p(t)$ is an essentially bounded non negative function with an essential bound equal to M. Then, there exists $\theta_0 > 1$ such that for all $\theta \geq \theta_0$, the function v satisfies

$$\forall t \geq 0, v(t) \leq e^{-\frac{ln(\theta)}{2\delta}(t-t_0)} \left(1 + 2b\delta \frac{\left(\sqrt{\theta}-1\right)}{ln(\theta)}\right) \max_{s\in[-\delta,0]} v(s) + \frac{2\delta M}{ln(\theta)} \tag{B.7}$$

Proof of the lemma B.3.1

Let consider the following Lyapunov function for $t \geq 0$,

$$W(t) = v(t) + b\sqrt{\theta} \int_{t-\delta}^{t} e^{-(t-s)\frac{ln(\theta)}{2\delta}} v(s) ds. \tag{B.8}$$

One gets

$$\begin{aligned}
\dot{W}(t) =& \dot{v}(t) - \frac{ln(\theta)}{2\delta}\left(W(t) - v(t)\right) + b\sqrt{\theta}v(t) - b\sqrt{\theta}v(t) - b\sqrt{\theta}e^{-\frac{ln(\theta)}{2}}v(t-\delta) \\
=& \dot{v}(t) - \frac{ln(\theta)}{2\delta}\left(W(t) - v(t)\right) + b\sqrt{\theta}v(t) - b\sqrt{\theta}v(t) - b\sqrt{\theta}v(t-\delta) \\
\leq& -\theta a v(t) - \frac{ln(\theta)}{2\delta}\left(W(t) - v(t)\right) + b\sqrt{\theta}v(t) + p(t)
\end{aligned} \tag{B.9}$$

The last inequality lead to

$$\dot{W}(t) + \frac{ln(\theta)}{2\delta} W(t) \leq -\theta \left(a - \frac{ln(\theta)}{2\theta\delta} - b\frac{1}{\sqrt{\theta}}\right) v(t) + p(t) \tag{B.10}$$

Since $a > 0$, it is clear that there exists $\theta_0 > 1$ such that for all $\theta \geq \theta_0$, one has

$$\left(a - \frac{ln(\theta)}{2\theta\delta} - b\frac{1}{\sqrt{\theta}}\right) \geq 0. \tag{B.11}$$

Hence, for $\theta > \theta_0$, one gets

$$\dot{W}(t) \leq -\frac{ln(\theta)}{2\delta} W(t) + p(t). \tag{B.12}$$

Using the comparison lemma, one gets

$$
\begin{aligned}
W(t) \leq & e^{-\frac{ln(\theta)}{2\delta}t}W(0) + \int_0^t e^{-(t-s)\frac{ln(\theta)}{2\delta}}p(s)ds \\
= & e^{-\frac{ln(\theta)}{2\delta}t}\left(v(0) + b\sqrt{\theta}\int_{-\delta}^{0} e^{s\frac{ln(\theta)}{2\delta}t}v(s)ds\right) + \frac{2\delta M}{ln(\theta)} \\
\leq & e^{-\frac{ln(\theta)}{2\delta}t}\left(1 + 2b\delta\frac{(\sqrt{\theta}-1)}{ln(\theta)}\right)\max_{s\in[-\delta,0]} v(s) + \frac{2\delta M}{ln(\theta)}
\end{aligned}
\tag{B.13}
$$

Combining the last inequality with the fact that $v(t) \leq W(t), \forall t \geq 0$ leads to (B.7). This ends the proof of the lemma B.3.1. ■

Lemma B.3.2 *Farza et al. (2018a) Consider a continuous real function* $x : t \in [-\delta, +\infty[\to x(t) \in [0, +\infty[$ *with* $\delta > 0$*, such that*

$$
x(t) \leq \rho e^{-at} + M + \eta\int_{t-\delta}^{t} e^{-a(t-s)}x(s)ds \ \ for \ \ t \geq 0, \tag{B.14}
$$

where $\rho, M \geq 0$ *and* $a, \eta > 0$ *are real numbers. If* $\eta\delta < 1$ *then, one has*

$$
x(t) \leq \frac{1}{1-\eta\delta}\left\{\left(\rho + \eta\int_{-\delta}^{0} x(s)ds\right)e^{-at} + M\right\}. \tag{B.15}
$$