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Diseño de observadores para sistemas algebro-diferenciales con parámetros variables

presentada por

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como requisito para la obtención del grado de

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Summary

Keywords: Algebro-differential systems, LPV systems, Generalized dynamic observer, Adaptive observer Lipschitz nonlinearities

In this thesis, the design of observers for nonlinear algebro-differential parameter-varying systems and their applications in fault estimation and parameter estimation are studied. Nonlinear algebro-differential parameter-varying systems allow the preservation of nonlinearities and a better understanding of the equations involved in the analyzed model, as well as a broader operating range.

The observer implemented in this work is known as the generalized dynamic observer (GDO). The core concept involves incorporating dynamics structure to augment its degrees of freedom, with the goal of achieving accuracy in steady-state and improve robustness in estimation error against disturbances and uncertainties in parameters.

The main idea is to use the advantages of this generalized structure for designing various methodologies in nonlinear algebro-differential parameter-varying systems, thus creating a more general framework than those documented in the literature.

Different estimation algorithms are presented to obtain diverse observer structures. The asymptotic stability of the observers is analyzed through Lyapunov analysis using Linear Matrix Inequalities (LMI), and the elimination lemma, which is employed to transform these LMIs while preserving the generalized structure of the observers

Finally, the performance of the presented observers is evaluated through some engineering applications to illustrate the effectiveness of the proposed approaches.

Resumen

Palabras clave: Sistemas algebro-diferenciales, Sistemas LPV, Observador dinámico generalizado, Observador adaptable, No linealidades Lipschitz

En esta tesis, se estudia el diseño de observadores para sistemas algebro-diferenciales no lineales con parámetros variables y sus aplicaciones en la estimación de fallas y la estimación de parámetros. Los sistemas algebro-diferenciales no lineales con parámetros variable permiten la preservación de las no linealidades y una mejor comprensión de las ecuaciones involucradas en el modelo analizado, así como un rango de operación más amplio.

El observador implementado en este trabajo se conoce como observador dinámico generalizado (GDO). El concepto central consiste en incorporar una estructura dinámica para aumentar sus grados de libertad, con el fin de lograr una precisión en estado estacionario y mejorar la robustez frente a errores de estimación debido a perturbaciones e incertidumbres parametricas

La idea principal es aprovechar las ventajas de esta estructura generalizada para diseñar diversas metodologías en sistemas algebro-diferenciales no lineales con parámetros variable, creando así un marco más general que el documentado en la literatura.

Se presentan varios algoritmos de estimación destinados a crear diversas estructuras de observadores. La estabilidad asintótica de los observadores se analiza a través del análisis de Lyapunov utilizando desigualdades matriciales lineales (LMIs), y se utiliza el lema de eliminación para transformar estas LMIs preservando la estructura generalizada de los observadores.

Por último, se evalúa el rendimiento de los observadores presentados a través de algunas aplicaciones en ingeniería, con el fin de ilustrar el desempeño de los enfoques propuestos.

With infinite love and gratitude, I dedicate this achievement to my beloved parents and siblings, whose unwavering support has been my greatest inspiration. Your love, encouragement, and sacrifices have been the driving force behind every step of this journey. Thank you for being my rock and guiding light throughout this thrilling adventure. And to my wife, your unwavering love and encouragement have been my source of strength and joy. I am forever grateful for your presence in my life

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Publications

Published paper

- Zetina-Rios, I. I., Osorio-Gordillo, G. L., Alma, M., Darouach, M., & Vargas-Méndez, R. A. State estimation strategy for a class of nonlinear algebro-differential parameter-varying systems. International Journal of Systems Science, 09 Dec 2023, 54(16), 3085-3097

Submited journal

- Zetina-Rios, I. I., Osorio-Gordillo, G. L., Alma, M., Darouach, M., Astorga-Zaragoza C. M."Fault estimation for a class of nonlinear algebro-differential parameter-varying systems. European Journal of Control", 2023.

Conference papers

- I. I. Zetina Rios G. L. Osorio Gordillo M. Alma M. Darouach C. M. Astorga Zaragoza "Fault estimation based on generalized learning observer for a class of nonlinear algebro-differential parameter-varying systems", Congreso Nacional de Control Automático, 25-27 de Octubre, 2023. Acapulco, Guerrero, México.
- I. I. Zetina Rios, M. Alma, G.-L. Osorio-Gordillo, M. Darouach, C. M. Astorga-Zaragoza. "Generalized adaptive observer design for a class of linear algebro-differential systems", 11th International Conference on Systems and Control (ICSC), December 18-20, 2023, Sousse, Tunisia.
- I. I. Zetina Rios, M. Alma, G. L. Osorio Gordillo, M. Darouach and C. M. Astorga Zaragoza "Parameter estimation based on generalized adaptive observer for a class of linear algebro-differential parameter-varying systems", 10th International Conference on Control, Decision and Information Technologies (CoDIT), Valletta, Malta on July 1-4, 2024.

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Notation and acronyms

Sets and norms

$\mathbb{R}$	Set of all real numbers.
$\mathbb{R}^n$	Set of n-dimensional real matrices.
$\mathbb{R}^{n\times m}$	Set of $n \times m$ dimensional real matrices.
$\mathbb{C}_+$	Complex half-plane open to the right
$\lvert a \rvert$	The absolute value of scalar a.
$\lVert a \rVert_1$	The norm-1 of signal a.

Notation related to vectors and matrices

$A > 0$	Positive definite matrix.
$A < 0$	Negative definite matrix.
I	Identity matrix of appropriate dimensions.
0	Matrix with zero elements of appropriate dimensions.
I_n	Identity matrix of dimension $n \times n$.
A^{-1}	Inverse of a matrix $A \in \mathbb{R}^{n\times n}, det(A) \neq 0$.
A^T	Transpose of a matrix A.
$A^\perp$	Orthogonal of matrix A.
A^+	Any generalized inverse of matrix A, verifying $AA^+A = A$.
$eig(A)$	Set of all eigenvalues of matrix A.
$det(A)$	Determinant of the matrix $A \in \mathbb{R}^{n\times n}$.
$rank(A)$	Rank of the matrix $A \in \mathbb{R}^{n\times m}$.
$(*)$	Transposed elements in symmetric position
$diag(A)$	Main diagonal of the matrix A.
$ones_{n,m}$	Matrix with ones of dimension $n \times m$.
$zeros_{n,m}$	Matrix with zeros of dimension $n \times m$.
$\mathcal{H}_e$	Matrix that is equal to its own conjugate transpose

Acronyms

CRAN	*Research Center for Automatic Control of Nancy.*
CENIDET	*National Center for Research and Technological Development.*
DOBC	*Disturbance Observer-based Controller.*
FAUIO	*Fast Adaptive Unknown Input Observer.*
GDO	*Generalize Dynamic Observer.*
GDLO	*Learning Generalized Dynamic Observer.*
GDNPVO	*Generalized Descriptor Nonlinear Parameter-Varying observer.*
IAE	*Integral of Absolute Error.*
ISE	*Integral Squared Error.*
ITAE	*Time-weighted Absolute Error.*
IFAC	*International Federation of Automatic Control.*
IO	*Interval Observer.*
LMI	*Linear Matrix Inequalities.*
LFT	*Linear Fractional Transformation.*
LTI	*Linear Time-Invariant.*
LTV	*Linear Time-Variant.*
LPV	*Linear Parameter-Varying.*
LO	*Learning Observers.*
NIO	*Non-Infinitely Observable.*
NN	*Neural Network.*
NLPV	*Nonlinear Parameter-Varying.*
PEMFC	*Fault Reconstruction of Proton Exchange Membrane Fuel Cell.*
QLIF	*Quasi-LPV Luenberger Interconnected Fuzzy.*
SMO	*Sliding Mode Observer.*
S-LPV	*Algebro-Differential Linear Parameter-Varying.*
S-NLPV	*Algebro-Differential Parameter-Varying.*
UBB	*Unknown-But-Bounded.*
UIO	*Unknown Input Observer.*

Chapter 1

General introduction

1.1 Context of the thesis

This report presents the results of the research thesis titled "Observers design for algebro-differential parameter varying systems". These results involve proposing a new observer structure for the estimation of state variables and faults for a class of nonlinear algebro-differential parameter-varying (S-NLPV) systems, and consequently, a structure of adaptive observers for a joint estimation of state and parameters for the same class of systems.

On the other hand, this research is carried out in collaboration between the National Center for Research and Technological Development (CENIDET) and the Research Center for Automatic Control of Nancy (CRAN) at Lorraine University.

1.2 Problem formulation

Based on the literature review, it was possible to identify some common problems and trends addressed in the literature, relying on the linear parameter-varying (LPV) polytopic approach and algebro-differential systems. However, there is an extensive literature on the design of observers for state, parameters and fault estimation in LPV systems and algebro-differential systems. Nevertheless, these cases could have some restrictions:

- In many practical cases, maintaining a nonlinear structure instead of reducing to a linear one can be more appropriate, and may lead to less conservativeness and over-approximation.
- Typically, the approaches to observer design presented in the literature are addressed through specific or limited structures.

These issues are still open problems and a possible contribution was found based on the following nonlinear algebro-differential parameter-varying systems:

$$
\begin{aligned}
E\dot{x}(t) &= \sum_{i=1}^{k} \mu_i(\rho(t))[A_i x(t) + B_i u(t)] + Df(t, F_L x) && (1.1) \\
y(t) &= \sum_{i=1}^{k} \mu_i(\rho(t))[C_i x(t) + D_i u(t)] && (1.2)
\end{aligned}
$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the known input, $y(t) \in \mathbb{R}^p$ is the measurement output vector. Matrices $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{p\times n}$ and $D_i \in \mathbb{R}^{p\times n}$ are known constant matrices and $f(t, F_L x)$ represents a nonlinearity that satisfies the Lipschitz constrain $||\Delta_f|| \leq \lambda||F_L(x_1 - x_2)||$ where $\Delta_f = f(t, F_L x_1) - f(t, F_L x_2)$, λ represents a known Lipschitz constant, while F_L is a real matrix of suitable dimensions.
Let $rank(E) = r < n$ and $E^{\perp} \in \mathbb{R}^{s\times n}$ be a full row matrix such that $E^{\perp}E = 0$, in this case $s = n-r$.

Considering $\mu_i(\rho(t))$ as membership functions constructed with predefined variant parameters$\rho(t) \in \mathbb{R}^l$. The membership functions have the following properties:

$$
\sum_{i=1}^{k} \mu_i(\rho(t)) = 1, \quad \mu_i(\rho(t)) \geq 0, \tag{1.3}
$$

for i ranging from 1 to k, where $k = 2^l$, representing the number of vertices in the polytope.

Most mathematical models inherently incorporate complex nonlinearities within their structure. The nonlinear function $f(t, F_L x)$ allows retaining the complete information of the original model in addition to the known advantages brought by the use of the LPV approach and algebro-differential systems.

1.3 Objectives of the thesis

1.3.1 General objective

Simultaneously estimate the state variables and non-measurable variable parameters in algebro-differential systems by designing an adaptive observer of unknown inputs using the variable parameter nonlinear systems approach.

1.3.2 Specific objectives

- Estimate non-measurable state variables in algebro-diferential systems using the nonlinear parameter-varying approach.
- Simultaneously estimate state variables and unknown inputs in algebro-diferential systems using the nonlinear parameter-varying approach.
- Design an adaptive observer of unknown inputs for the estimation of state variables with non-measurable parameters in nonlinear algebro-differential parameter-varying approach.

1.4 Justification

The precise estimation of state variables in nonlinear systems is crucial for achieving efficient control and monitoring in various control applications. Nevertheless, the use of nonlinear approaches for different control techniques poses significant challenges due to its inherent complexity.

In the last decades, the LPV framework has proved to be suitable for addressing nonlinear control problems, since many nonlinearities can be “embedded” in the varying parameters, thus transforming the nonlinear system into an equivalent quasi-LPV representation. However, in many cases, maintaining a nonlinear structure instead of reducing to a linear one can be more appropriate, and may lead to less conservativeness and over-approximation (Sename and Rotondo, 2021).

The nonlinear parameter-varying (NLPV) systems are capable to maintain certain nonlinearities of the original nonlinear system, which has been a positive topic trend in new research. On the other hand, algebro-differential systems (known as generalized, singular, or descriptor systems) are a special class of systems governed by both dynamic and algebro equations and can represent a broader range of systems such as electrical circuits, biological systems, mechanics, and chemical processes, among others (Zhang et al., 2019).

This research aims to contribute to the design of observers for state estimation, faults with unknown inputs, and unknown parameters in nonlinear algebra-differential parameter-varying systems. There is a vast literature on these topics, but there are still issues that have not been addressed.

1.5 Outline of the thesis

The following chapters are organized as shown below:

In **Chapter 2**, an updated bibliographic review is presented for LPV and NLPV systems, algebro-differential systems, and adaptable observers. This is done to highlight the opportunity gap addressed in this thesis work.

In **Chapter 3**, fundamental concepts are presented regarding algebro-differential systems, LPV, and quasi-LPV. Their classification, examples, and important properties for this type of systems are discussed. Subsequently, observers for algebro-differential systems, LPV, adaptable observers, and unknown input observers are introduced.

In **Chapter 4**, a generalized dynamic observer is introduced for nonlinear algebro-differential systems with parameter-varying to estimate state variables. Sufficient conditions for the existence of the observer which guarantee stability regarding Linear Matrix Inequalities constraints using the Lyapunov stability theory are given. The approach presented includes the proportional and proportional-integral observer as particular cases and allows some robustness concerning uncertainties and modelling errors and its steady state accuracy. A rolling disc model is given to illustrate our results.

In **Chapter 5**, the design and analysis of a generalized observer for the simultaneous estimation of state variables and unknown inputs are conducted for nonlinear algebro-differential parameter-varying systems. Also, the design of a generalized learning observer (GLO) structure for the simultaneous estimation of variable states and actuator faults is presented for the same class of systems. The generalized observer structure allows robustness and steady state accuracy in simulation. The conditions of existence and stability are given in terms of LMIs. The obtained results are illustrated on the heat exchanger with two countercurrent cells model with unknown inputs.

In **Chapter 6**, an adaptive observer design is presented for the simultaneous estimation of system parameters and state variables for a class of linear algebro-differential systems. Also, an observer for simultaneous estimation of parameters and state variables for a category of algebro-differential linear parameter-varying (S-LPV) systems by employing a generalized adaptive observer is presented. Sufficient conditions for the existence of the observer, ensuring stability with respect to Linear Matrix Inequalities (LMIs) constraints, are provided. The methodology presented is a generalization of proportional and proportional-integral observers and It allows for robustness with respect to uncertainties and modeling errors, as well as steady-state accuracy. A numerical example is given to illustrate our results.

Finally, in **Chapter 7**, some concluding remarks of this research work are presented and some perspectives for the future work are discussed.

Chapter 2

Bibliography review

In this section, an updated and detailed bibliographic review of various crucial topics for the thesis is conducted. The importance of each topic is explored, its current trends are analyzed, and future developments are projected. The review contextualizes the research, identifies essential connections, and highlights areas of academic focus. This analysis not only provides background but also points towards possible future directions in the field.

2.1 Analysis for LPV and NLPV observers

The LPV methodology has been widely used in recent years. Numerous researchers have studied this methodology because many physical systems can be modeled as LPV systems. This allows reducing the complexity of the analysis, as many of the well-established techniques used in linear systems can be extended to LPV systems (Abdullah and Zribi, 2009).

In many control applications, have knowledge of all the state variables is necessary, although this is not always possible. Almost two decades ago, LPV techniques emerged as an efficient solution to address a wide operating range without the need for discretization. They provide reliability in terms of stability and performance in the face of parameter variations, standing out for their operational simplicity (Mohammadpour and Scherer, 2012). In previous years, the principles of observer design for LPV systems have been established. First introduced by Shamma (1988), followed by some relevant works such as in G. Iulia Bara and Ragot (2001); Astorga-Zaragoza et al. (2011); Estrada et al. (2015b); Pérez Estrada et al. (2017).

LPV observers

Several relevant works in more recent years have conducted significant studies on the observer designs of LPV systems. Oliveira and Pereira (2019) presents an unknown input proportional-integral observer (UIO) that makes use of an explicit integration term for discrete LPV systems. In Agulhari and Lacerda (2019), conditions for the observer-based control problem for LPV periodic discrete-time systems are presented, and an H_∞ guaranteed cost has been employed to provide robustness against external noises. A polytopic LPV observer is designed to estimate the damper force by utilizing the dynamic nonlinear model of the electrorheological (ER) damper in a quarter-car system. The system is represented in LPV form, considering a phenomenological model of the damper presented in Pham et al. (2019). In Díaz et al. (2021), an observer design for discrete time-varying descriptor systems is presented. It is assumed that the dynamical system matrix is a function of time-varying parameters that are not precisely known.

Reference Yang et al. (2020) presents an augmented LPV observer for the high-precision state estimation and fault reconstruction of proton exchange membrane fuel cell (PEMFC) air management system. In Fouka et al. (2021) a Quasi-LPV Luenberger interconnected fuzzy observer (QLIF) synthesis for simultaneous estimation of the lateral and longitudinal vehicle dynamics. The outlined observer is designed considering Quasi-LPV vehicle interconnected model taking into account real con- straints such as the variations in the immeasurable longitudinal and lateral speed, non-linearities of steering angle and the tire slip velocities during the interconnected-sub observers design. Reference Lamouchi et al. (2022) presents a LMIs formulation to design an active fault tolerant control for polytopic uncertain LPV systems subject to uncertainties and actuator faults.

NLPV observers

It has been shown that in many instances, maintaining a nonlinear structure instead of reducing it to a linear one can be more appropriate and may lead to less conservatism and over-approximation Sename and Rotondo (2021). The nonlinear Parameter Varying systems are capable to maintain certain nonlinearities of the original nonlinear system, which has been a positive topic trend in new research. In Hassan et al. (2014), the autors consider the design of an observer-based controller for time-delay systems with Lipschitz nonlinearities. A nonlinear observer design of a diesel engine model is presented in Boulkroune et al. (2015), the observer is developed for a general class of nonlinear systems with a locally or globally bounded Jacobian.

The robust full-order and reduced-order observers design for LPV systems with one-sided Lipschitz nonlinearities and disturbances are addressed in Abdullah and Qasem (2019). In Pham et al. (2021) two NLPV classical observers were presented to estimate the damper force, utilizing a dynamic nonlinear model. In Zhu et al. (2022), the disturbance observer-based controller (DOBC) design problem for a class of uncertain NPV systems subject to unknown uncertainty and unmeasurable state variables has been investigated. Reference Zhang and Liu (2022) introduces the set-membership estimation problem in the context of discrete NLPV. Additionally, a sufficient condition for set-membership estimation is derived, taking into consideration unknown-but-bounded (UBB) noise.

Algebro-differential observers

Due to the fact that the algebro-differential system can represent a wider range of systems, such as electrical circuits, biological systems, mechanics, and chemical processes, among others complex systems, several works have been carried out in recent years Zhang et al. (2019). A generalized dynamic observer for descriptor systems is synthesized to perform a convergent estimation of actuator faults is presented in Osorio-Gordillo et al. (2018). Reference Ríos-Ruiz et al. (2019) presents a finite-time convergent functional dynamical observer design for descriptor systems applied to the sensor and actuator faults detection and estimation. An interval observer (IO) for discrete-time descriptor systems with uncertainties is presented in Liu et al. (2023).

Authors in Do et al. (2020) present a H_∞ observer with parameter-dependent stability to attenuate the disturbance impact on estimation error for singular LPV systems (S-LPV). In reference Zhang et al. (2023) is presented a sliding mode observer (SMO) design method to estimate the states and unknown inputs (UIs) in a class of non-infinitely observable (NIO) descriptor systems that contain UIs in both the state and output equations. In Zetina-Rios et al. (2023) a GDO for descriptor nonlinear parameter varying systems is presented to deal with state estimation in a rolling disc

Adaptive observers

Since adaptive observers are capable of estimating the vector of state variables and unknown parameters, significant work has been carried out in recent years to address these issues. In the literature, two primary approaches have been developed for the design of adaptive observers. These approaches are essentially grounded on the following principles:

1. The unknown parameter vector is inferred from the stability analysis of a state observer, and the convergence property of the parameter error is achieved through the persistence of excitation-type constraint. Consequently, a parameter adaptation law is proposed
2. Through an augmented system for which the adaptive observer design is elaborated. In this case, the system dynamics are augmented with the dynamics of its unknown parameters.

For the first case, relevant works have been presented in recent years. Reference Gaudio et al. (2021) introduces a parameter estimation algorithm for the adaptive control of a class of time-varying plants with a matrix of time-varying learning rates. This approach enables parameter estimation error trajectories to converge exponentially fast towards a compact set whenever excitation conditions are satisfied. An adaptive neural network (NN) optimized output-feedback control problem is studied for a class of stochastic nonlinear systems with unknown nonlinear dynamics, input saturation, and state constraints is presented in Li et al. (2022). In reference Alma et al. (2023) presents the adaptive observer design for a class of nonlinear descriptor systems for simultaneous estimation of the system states and its parameters.

For the second case, innovative works have been recorded in recent years. In Chen et al. (2019), the problem of fault observer design is investigated for Markovian jump systems with simultaneous time-varying actuator efficiency factors, as well as additive faults in actuators and sensors. Reference Zhang et al. (2020) proposes a state augmentation approach to achieve interval fault estimation for descriptor systems with unknown but bounded disturbances and measurement noises. A Fast Adaptive Unknown Input Observer (FAUIO) is proposed in Gao et al. (2022) to enhance the fault estimation performance of the system.

2.2 Conclusion

After reviewing and analyzing the literature, it is evident that the design of observers for LPV systems has been extensively explored, as indicated in Table 2.1. Subsequently, an extension to algebro-differential systems is presented. However, there exists an implementation gap in the development of methodologies for algebraic-differential NLPV systems. This is due to the fact that maintaining nonlinearities along with the aforementioned methodologies expands the knowledge spectrum of the original model and reduces conservatism for the application of more complex and precise control techniques. Additionally, it is possible to consider methodologies for fault estimation and unknown inputs applied to this type of systems. Furthermore, the application of adaptable observers allows for the proper estimation of unknown system parameters. An important contribution in this thesis work is the design of adaptable observers applied to this type of systems.

Table 2.1. Bibliography review table

LPV observers			
Reference	Observer	System	Description
Oliveira and Pereira (2019)	UIO	Discrete LPV	Makes use of an explicit integration term for discrete LPV systems.
Agulhari and Lacerda (2019)	H_∞	Discrete LPV	Provide robustness against external noises.
Yang et al. (2020)	Augmented LPV	PEMFC	Presents a high-precision state estimation and fault reconstruction
Díaz et al. (2021)	Classical	Discrete time-varying descriptor	An affine parameter-dependent Lyapunov function is considered to ensure the error convergence
Fouka et al. (2021)	QLIF	Quasi-LPV	Simultaneous estimation of the lateral and longitudinal vehicle dynamics
NLPV observers			
Hassan et al. (2014)	Observer-based controller	Time-delay systems	For state estimation
Boulkroune et al. (2015)	Nonlinear	NLPV systems	Developed for a general class of nonlinear systems with a locally or globally bounded Jacobian
Abdullah and Qasem (2019)	Full and reduced-order observers	NLPV systems	One-sided Lipschitz nonlinearities class of nonlinear systems and disturbances are addressed
Pham et al. (2021)	Clasical observers	NLPV systems	Estimate the damper force
Zhu et al. (2022)	Observer-based controller	NPV systems	Unknown uncertainty and unmeasurable state variables
Algebro-differential observers			
Osorio-Gordillo et al. (2018)	GDO	Descriptor systems	Convergent estimation of actuator faults is presented
Ríos-Ruiz et al. (2019)	Functional dynamical	Descriptor systems	actuator faults detection and estimation
Do et al. (2020)	H_∞	Descriptor systems	Presents an observer with parameter-dependent stability
Liu et al. (2023)	Interval observer	Discrete systems	Estimation of uncertainties
Zhang et al. (2023)	Sliding mode	Descriptor systems	Is a method to estimate the states and unknown inputs

Chapter 3

Theoretical framework

The present chapter is dedicated to the description of different concepts and definitions about algebro-differential systems, LPV systems and observers.

In Section 3.1, algebra-differential systems are described, along with the mathematical derivation of some examples. In Section 3.2, LPV systems and their representation are addressed. Likewise, quasi-LPV systems and NLPV Systems are defined. Finally, in Section 3.3, definitions of observers are presented, along with some types of observers reported in the literature.

3.1 Algebro-differential systems

The state space approach was developed in the late 1950s and early 1960s. This approach offers significant advantages as it provides an efficient method for the analysis and synthesis of control systems. Additionally, it offers a deeper understanding of the various properties of systems.

State space models of systems are primarily obtained using the state space variable method. To obtain a state space model of a practical system, it is necessary to select some physical variables such as speed, weight, temperature, or acceleration, which are sufficient to characterize the system.

Then, by the physical relationships among the variables or through model identification techniques, a set of equations can be established. Typically, this set of equations consists of differential and/or algebraic equations, forming the mathematical model of the system. By properly defining a state vector $x(t)$ and an input vector $u(t)$, composed of the physical variables of the system, and an output vector $y(t)$, whose elements are appropriately chosen measurable variables of the system, this set of equations can be organized into two equations. One of them is the so-called state equation, which follows the general form:

$$f(\dot{x}, x(t), u(t), y(t), t) = 0, \tag{3.1}$$

and the other is the output equation, or the observation equation, which is in the form of

$$g(x(t), u(t), y(t), t) = 0, \tag{3.2}$$

where f and g are linear vector functions of appropriate dimensions with respect to $\dot{x}(t)$, $x(t)$, $u(t)$, $y(t)$ and t. Equations (3.1) and (3.2) give the state space representation for a general nonlinear dynamical system.

The following equation describes a special form of (3.1) and (3.2)

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{3.3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the known input, $y(t) \in \mathbb{R}^p$ is the measurement output vector. Matrix $E \in \mathbb{R}^{n\times n}$ could be singular. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$ are real matrices.

3.1.1 Examples of algebro-differential systems

Algebro-differential systems appear in many fields, such as power systems, electrical networks, aerospace engineering, chemical processes, social economic systems, network analysis, biological systems, time-series analysis, and so on. In this section, some examples of descriptor systems are presented, from which readers can indeed see the existence of descriptor linear systems in our real world.

Electrical circuit Systems

Numerous electrical circuit systems can be characterized using descriptor linear systems.

Example 1. *(Dai, 1987) Consider a simple circuit network as shown in Figure 3.1, where R, L and C stand for the resistor, inductor, and capacity, respectively, and their voltages are denoted by $V_R(t)$, $V_L(t)$, $V_C(t)$ respectively. $V_S(t)$ is the voltage source which is taken as the control input. $I(t)$ is the current total in the circuit. Following basic circuit theory and the Kirchoff's law (Smith and Dorf, 1992), we have the following equations, which describe the system:*

$$L\dot{I}(t) = V_L(t), \tag{3.4}$$

$$\dot{V}_C(t) = \frac{1}{C} I(t), \tag{3.5}$$

$$R_1 I(t) = V_{R1}(t), \tag{3.6}$$

$$R_2 I(t) = V_{R2}(t), \tag{3.7}$$

$$V_L(t) + V_C(t) + V_R(t) = V_S(t). \tag{3.8}$$

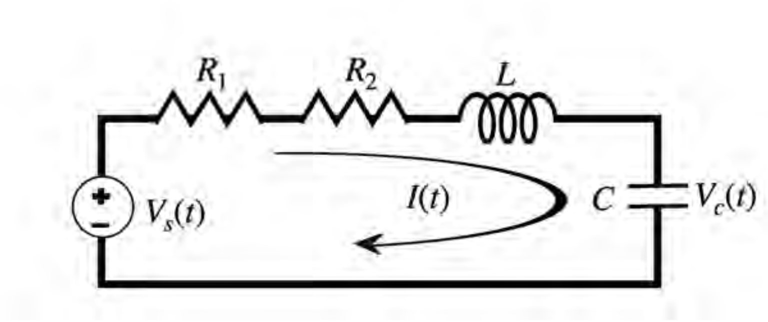

Fig. 3.1. A single-loop circuit network.

Realizing the change of variable

$x_1(t) = I(t)$, $x_2(t) = V_L(t)$, $x_3(t) = V_c(t)$, $x_4(t) = V_{R1}(t)$ *and* $x_5(t) = V_{R2}(t)$

Equations (3.4)–(3.8), *with the change of variable taken into account, can be written in the following descriptor linear system form:*

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{3.9}$$

with $E = \begin{bmatrix} L & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \dfrac{1}{C} & 0 & 0 & 0 & 0 \\ -R_1 & 0 & 0 & 1 & 0 \\ -R_2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$,
$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ *and* $y = \begin{bmatrix} x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix}$.

Where the rank$(E) = r < n$, and let $E^{\perp} \in \mathbb{R}^{s \times n}$ be a full row matrix such that $E^{\perp}E = 0$. In this case, $s = n - r$, resulting in a singular matrix.

Mechanical systems

Example 2. *Constrained linear mechanical systems can be described as follows:*

$$M\ddot{z}(t) + D\dot{z}(t) + Kz(t) = Lf(t) + J\mu(t), \tag{3.10}$$
$$Gz(t) + Hz(t) = 0, \tag{3.11}$$

where $z(t) \in \mathbb{R}^n$ is the displacement vector, $f(t) \in \mathbb{R}^n$ is the vector of known input forces, $\mu(t) \in \mathbb{R}^q$ is the vector of Lagrangian multipliers, M is the inertial matrix, which is usually symmetric and positive definite, D is the damping and gyroscopic matrix, K is the stiffness and circulator matrix, L is the force distribution matrix, J is the Jacobian of the constraint equation, G and H are the coefficient matrices of the constraint equation. All matrices in (3.10)–(3.11) are known and constant ones of appropriate dimensions.
Equation (3.10) is the dynamical equation, while (3.11) is the constraint equation. Assume that a linear combination of displacements and velocities is measurable, then, the output equation is of the form

$$y(t) = C_p z(t) + C_v \dot{z}(t), \tag{3.12}$$

where C_p, $C_v \in \mathbb{R}^{m \times n}$. By further choosing the state vector and the input vector as $x(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \\ \mu(t) \end{bmatrix}$ *and* $u(t) = f(t)$,

respectively, then the above (3.10)-(3.12) can be written as (3.9) with $E = \begin{bmatrix} I & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

$A = \begin{bmatrix} 0 & I & 0 \\ -K & -D & J \\ H & G & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ L \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} C_p & C_v & 0 \end{bmatrix}$ *and* $y = \begin{bmatrix} C_p x_1(t) + C_v x_2(t) \end{bmatrix}$.
Where the rank$(E) = r < n$, and let $E^{\perp} \in \mathbb{R}^{s \times n}$ be a full row matrix such that $E^{\perp}E = 0$. In this case, $s = n - r$, resulting in a singular matrix.

3.1.2 Algebro-differential system properties

Definition 1. *(Yip and Sincovec, 1981): The* ***regularity*** *property in descriptor systems ensures the existence and uniqueness of solutions. System (3.3) is considered regular if there exists a constant scalar $s \in C$ such that*

$$det(sE - A) \neq 0 \tag{3.13}$$

or equivalently, the polynomial $det(sE - A)$ is not identically zero. In this context, we also refer to the pair (E, A) or the matrix pencil $sE - A$ is regular.

Definition 2. *(Duan, 2010): A descriptor linear system (3.3) is deemed* ***impulse-free*** *if, when starting from any arbitrary initial value, its state response lacks impulse terms.*

$$rank \begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = n + rank(E) \tag{3.14}$$

Definition 3. *The pair (E, A) of system (3.3) is said to be* ***admissible if it is regular, impulse-free, and stable****.*

The following condition of system (1.2) is a generalization of the impulse observability

$$rank \begin{bmatrix} E & A \\ 0 & C \\ 0 & E \end{bmatrix} = rank(E) + n, \tag{3.15}$$

Impulsive terms are not desirable since they can saturate the state response. Impulse-observability (I-observability) guarantees the ability to estimate impulse terms given by the algebraic equations.

Definition 4. *The ability to reconstruct only the reachable state from the output data is characterized by the* ***Reachable-observability*** *(R-observability).*
The algebro-differential system of system (1.2) *is R-observable if and only if*

$$rank \begin{bmatrix} sE - A \\ C \end{bmatrix} = n, \forall s \in \mathbb{C}, \quad s \text{ finite.}$$

3.1.3 LPV systems

Linear parameter-varying (LPV) systems represent a fundamental class of dynamic systems that play a crucial role in a wide range of applications. These systems offer the flexibility to adapt to changes in the environment or operating conditions, making them a valuable tool in fields ranging from control engineering and robotics to aeronautics and biology.

The framework LPV systems concerns linear dynamical systems whose state-space representations depend on exogenous non-stationary parameters, as in

$$\begin{aligned} \dot{x}(t) &= A(\theta(t))x(t) + B(\theta(t))u(t), \\ y(t) &= C(\theta(t))x(t), \end{aligned} \tag{3.16}$$

where $u(t)$ is the input, $x(t)$ is the state variable, $y(t)$ is the output, and $\theta(t)$ is an exogenous parameter that can be time-dependent.
The terminology "linear parameter-varying" was introduced in Shamma (1988) to distinguish LPV systems from both LTI (linear time-invariant) and LTV (linear time-varying) systems. The distinction from LTI systems is clear in that LPV systems are non-stationary. The distinction from LTV systems is less evident, given that for any trajectory of the parameter $\theta(\cdot)$, the dynamics of (3.16) constitute a linear time-varying system. Instead, LPV systems are set apart from LTV systems by the perspective adopted in both analysis and synthesis.

3.1.4 Representation of LPV systems

An LPV system can be categorized into different groups depending on the role of parameters in the system equations. Essentially, there are three comprehensive formulations for LPV systems. (Briat, 2014):

- Polytopic formulation.

- Parameter-dependent formulation.
- Formulation for a Linear fractional transformation (LFT).

In the course of developing this thesis, the focus was on investigating LPV systems, particularly emphasizing their polytopic representation. This choice was made because the polytopic representation stands out as a widely utilized approach in the existing literature on LPV systems.

Polytopic formulation

The polytopic formulation of LPV systems involves representing them using polytopes, geometric figures with flat sides in multi-dimensional space. This approach describes the LPV system's behavior through a set of linear equations or inequalities associated with distinct regions in the parameter space, each corresponding to a facet or vertex of the polytope. This representation enables a piecewise-linear approximation of the system's dynamics, offering a convenient method for analyzing and controlling LPV systems with varying parameters.

The polytopic representation of an LPV system involves formulating weighting functions, allowing for the determination of system variables as the cumulative sum of the models specified at each vertex of the polytope. Each weighting function must satisfy the following conditions:

$$0 \leq \mu_i(\rho(t)) \leq 1, \ \sum_{i=1}^{2^r} \mu_i(\rho(t)) = 1 \tag{3.17}$$

where $\mu_i(\rho(t))$ s the parameter-dependent weighting function. The parameter vector $\rho(t)$ lies within a polytope with 2^r vertices, where r is the number of varying parameters. Each vertex of the polytope is formed by the combination of the extreme values of each parameter.

The general form of an LPV system with a polytopic formulation is as shown below.

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{k} \mu_i(\rho(t))[A_i x(t) + B_i u(t)] \\ y(t) &= C x(t) \end{aligned} \tag{3.18}$$

where $x(t) \in \mathbb{R}^n$is the state vector of the system, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^p$ is the vector of measurable outputs, A_i, B_i and C are known real matrices of appropriate dimensions, $\rho(t) \in \mathbb{R}^r$ is the vector of r varying parameters. Each of the k combinations of the variable parameter limits is evaluated, thus generating a set of local models, where $k = 2^r$, with r equal to the number of varying parameters.

3.1.5 Quasi-LPV Systems

When LPV systems are obtained by considering the nonlinearities of the model, the weighting functions become functions of the system's state. This particular type of LPV systems is known as quasi-LPV systems (Briat, 2014).

As an example, consider the following nonlinear system:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= x_1^3\end{aligned} \tag{3.19}$$

which can be represented as

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= \varrho(t)x_1(t)\end{aligned}$$

where $\varrho(t) = x_1^2 \in \mathbb{R}$.

It is noticeable that the nonlinearity can be expressed through a quasi-LPV approach, enabling the transformation of the nonlinear equation into a linear form. Given that $\varrho(t)$ represents a varying parameter with known variation limits but an unknown trajectory, the weighting functions depend on the variations of $\varrho(t)$, which, in turn, change based on the dynamics of the system's states.

The general form of a quasi-LPV system in its polytopic form is as follows:

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{k} \mu_i(\varrho(t))(A_i x(t) + B_i u(t))\\ y(t) &= Cx(t)\end{aligned} \tag{3.20}$$

where $x(t) \in \mathbb{R}^n$ is the state vector of the system, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^p$ is the vector of measurable outputs, A_i, B_i, and C are known real matrices, $\varrho(t) \in \mathbb{R}^r$ is the vector of r varying parameters **including the dynamics of the state variables**, and $\mu_i(\varrho(t))$ are the weighting functions.

3.1.6 Nonlinear parameter-varying systems

Frequently, "linearizing" nonlinear systems using the LPV language, known as the quasi-LPV representation, comes with the cost of reducing the generality of the system representation. Recall that to be assigned as a scheduling parameter, a (nonlinear) function of the state $x(t)$ must be known or estimated and bounded at least in the region where $x(t)$ remains. This condition, even if it can be satisfied, would certainly increase conservatism.

Often, by strategically choosing to maintain a certain level of nonlinearity in the system representation (rather than render it linear thanks to the LPV technique), we can benefit from interesting properties of nonlinear functions, e.g., Lipschitz conditions, which reduce conservatism and lead to more realistic results. This idea leads to the so-called nonlinear parameter varying (NLPV) class of systems that has emerged as a potential research trend in the LPV community (Sename and Rotondo, 2021). A NLPV system can be described by the following equations :

$$\dot{x}(t) = \sum_{i=1}^{k} \mu_i(\rho(t))[A_i x(t) + B_i u(t)] + Df(t, FLx, u) \tag{3.21}$$

$$y(t) = Cx(t) + Du(t) \tag{3.22}$$

where $x(t) \in \mathbb{R}^n$ is the semi–state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^p$ is the output vector and $f(t, F_L x)$ is the nonlinearity that verifies the Lipschitz constrain.

3.1.7 Observers for NLPV systems

Consider the nonlinear algebro-diferential parameter-varying system in its polytopic form as in (3.21) - (3.22) and considering the membership functions presented in (3.17). The following proportional observer is proposed.

$$\dot{\zeta}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[N_i\zeta(t) + F_i y(t) + TD_i f(t, F_L\hat{x}) + J_i u(t)\Big], \tag{3.23}$$

$$\hat{x}(t) = P\zeta(t) + Qy(t), \tag{3.24}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer, and $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$. The matrices $N_i \in \mathbb{R}^{q_0 \times q_0}, F_i \in \mathbb{R}^{q_0 \times p}, J_i \in \mathbb{R}^{q_0 \times m}, P \in \mathbb{R}^{n \times q_0}, Q \in \mathbb{R}^{n \times p}$ and $T \in \mathbb{R}^{q_0 \times n}$ are unknown matrices of appropriate dimensions, which must be determined such that $\hat{x}(t)$ converges asymptotically to $x(t)$.

This type of observer allows the inclusion of time-varying terms and nonlinearities within its structure. Retaining nonlinearities in a system can be beneficial for several reasons:

- **More accurate representation**: Many systems in reality are inherently nonlinear. By preserving nonlinearities in the model, a more accurate representation of the system's behavior is achieved.

- **Modeling complex phenomena**: Some phenomena in real systems can be more accurately modeled using nonlinear functions. Nonlinearity allows capturing complex behaviors that cannot be effectively described with linear models.

- **Adaptability to changes**: Nonlinear systems are often more adaptable to variations in operating conditions. They can better handle disturbances and fluctuations compared to linear systems, making them more robust in certain scenarios.

- **Performance improvement**: In some cases, nonlinearities can enhance the performance of the system. For example, in control, introducing nonlinearities can help improve stability, dynamic response, and reference tracking capability.

- **Diversity of behaviors**: Nonlinear systems can exhibit a wide variety of behaviors, including bifurcations and chaotic phenomena. This can be useful in applications where there is a desire to explore and leverage this diversity for specific purposes.

It is important to note that, although nonlinearities can offer advantages in representation and performance, they can also complicate the analysis and control design. The choice to retain or eliminate nonlinearities in a system often depends on specific modeling and control objectives.

3.1.8 Generalized observer

Recently, the study of a new observer structure, called the Generalized Dynamic Observer (GDO), has been introduced. These observer structures are based on Park et al. (2002) and Marquez (2003), where the principal idea is to add dynamic structure to increase their degrees of freedom, achieve steady-state accuracy and improve robustness in estimation error against disturbances and parametric uncertainties. Therefore, this structure can be considered more general than Proportional Observers (PO) and Proportional-Integral Observers (PIO).

Considering the following linear system

$$\begin{aligned} \dot{x}(t) =& Ax(t) + Bu(t) \\ y(t) =& Cx(t) \end{aligned} \tag{3.25}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the known input and $y(t) \in \mathbb{R}^p$ is the measurement output vector. Matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are known constant matrices.
The GDO structure for system (3.25) is described by

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + Fy(t) + Ju(t)] \tag{3.26}$$

$$\dot{v}(t) = S\zeta(t) + Lv(t) + My(t)] \tag{3.27}$$

$$\hat{x}(t) = P\zeta(t) + Qy(t) \tag{3.28}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents states vector of the observer, $v(t) \in \mathbb{R}^{q_1}$ is the auxiliary vector $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$. Matrices $N \in \mathbb{R}^{q_0\times q_0}$, $H \in \mathbb{R}^{q_0\times q_1}$, $F \in \mathbb{R}^{q_0\times p}$, $S \in \mathbb{R}^{q_1\times q_0}$, $L \in \mathbb{R}^{q_1\times q_1}$, $M \in \mathbb{R}^{q_1\times p}$, $P \in \mathbb{R}^{n\times q_0}$, $Q \in \mathbb{R}^{n\times p}$, and $J \in \mathbb{R}^{q_0\times m}$ are unknown matrices of appropriate dimensions that must be determined such that $\hat{x}(t)$ converge to $x(t)$.

3.1.9 Adaptive observers

In the design of observers, it is often assumed that all the parameters of the system are known. However, this is not always true. Adaptive observers offer an effective solution to this challenge, enabling the estimation of state variables along with system parameters. This is why they have been widely used in the literature.

Let us consider the following linear time invariant descriptor system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + \psi(t)\theta(t), \\ y(t) &= Cx(t), \end{aligned} \tag{3.29}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the known input, $y(t) \in \mathbb{R}^p$ is the measurement output vector and $\theta(t) \in \mathbb{R}^l$ is the unknown parameter vector assumed to be constant. Matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are known constant matrices. $\psi(t) \in \mathbb{R}^{m\times l}$ is a matrix of known signals, and it is assumed to be piecewise differentiable.
Let us consider the following adaptive observer for system (3.29)

$$\begin{aligned}
\dot{\zeta}(t) &= N\zeta(t) + Fy(t) + Ju(t) + T\psi\hat{\theta}(t) + \Upsilon\dot{\hat{\theta}}(t), \\
\hat{x}(t) &= P\zeta(t) + Qy(t), && (3.30) \\
\dot{\hat{\theta}}(t) &= C^T \Lambda(y(t) - C\hat{x}(t)), && (3.31)
\end{aligned}$$

where $\hat{x}(t) \in \mathbb{R}^n$ and $\theta(t) \in \mathbb{R}^l$ are the estimates of $x(t)$ and $\theta(t)$, respectively. The matrices $N \in \mathbb{R}^{q_0 \times q_0}$, $F \in \mathbb{R}^{q_0 \times p}$, $J \in \mathbb{R}^{q_0 \times m}$, $P \in \mathbb{R}^{n \times q_0}$, $Q \in \mathbb{R}^{n \times p}$ and $T \in \mathbb{R}^{q_0 \times n}$ are unknown matrices of appropriate dimensions, which must be determined such that $\hat{x}(t)$ converges asymptotically to $x(t)$ and $\hat{\theta}(t)$ converges to $\theta(t)$, respectively.

3.1.10 Unknown input observer

In the context of control systems, it's crucial to recognize that a predominant number of physical processes face disturbances in the form of measurement noises, modeling uncertainties, sensor irregularities, and faults in actuators. These disturbances collectively constitute what we refer to as unknown inputs. Managing and mitigating the impact of these unknown inputs is a critical aspect of designing robust control strategies for optimal system performance.
Let us consider the following linear time invariant descriptor system:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + Dd(t) + Ew(t), && (3.32) \\
y(t) &= Cx(t),
\end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the known input, $w(t) \in \mathbb{R}^{n_w}$ represents the disturbance vector, $d(t) \in \mathbb{R}^{n_d}$ is the unknown input vector and $y(t) \in \mathbb{R}^p$ Matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$ are known constant matrices.

In practice, often the disturbances or partial inputs are not known, or can result from either model uncertainties or faults. An effective way to address this issue is by designing an observer for a system similar to (3.32).

3.1.11 Tools for the stability analysis of dynamic systems

The following lemmas are used in the sequel of the document.

Lemma 1. *(Schur complement) Let A, B and D be matrices of appropriate dimension. Then the following statements are equivalent :*

(i) $\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} < 0.$

(ii) $D < 0$ *and* $A - BD^{-1}B^T < 0.$

(iii) $A < 0$ *and* $D - B^T A^{-1} B < 0.$

Lemma 2. *(Skelton et al., 1997) Let matrices $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be given. Then, the following statements are equivalent:*
(i) There exist a matrix Φ satisfying

$$\mathcal{B}\Phi\mathcal{C} + (\mathcal{B}\Phi\mathcal{C})^T + \mathcal{D} < 0.$$

(ii) The following two conditions hold

$$\begin{aligned} \mathcal{B}^{\perp}\mathcal{D}\mathcal{B}^{\perp T} < 0 \quad \text{or} \quad \mathcal{B}\mathcal{B}^T > 0. \\ \mathcal{C}^{T\perp}\mathcal{D}\mathcal{C}^{T\perp T} < 0 \quad \text{or} \quad \mathcal{C}^T\mathcal{C} > 0. \end{aligned}$$

Suppose that the statement (ii) holds, then $(\mathcal{C}_l,\, \mathcal{C}_r)$ and $(\mathcal{B}_l, \mathcal{B}_r)$ are any full rank factors of $\mathcal{B}$ and $\mathcal{C}$ such that $\mathcal{B} = \mathcal{B}_l\mathcal{B}_r$ and $\mathcal{C} = \mathcal{C}_l\mathcal{C}_r$ are verified.

Then the matrix Φ in statement (i) is given by

$$\Phi = {\mathcal{B}_r}^+\mathcal{K}_i{\mathcal{C}_l}^+ + \mathcal{Z} - {\mathcal{B}_r}^+\mathcal{B}_r\mathcal{Z}\mathcal{C}_l{\mathcal{C}_l}^+,$$

where

$$\begin{aligned} &\mathcal{K} = -\mathcal{R}^{-1}{\mathcal{B}_l}^T\mathcal{V}{\mathcal{C}_r}^T(\mathcal{C}_r\mathcal{V}{\mathcal{C}_r}^T)^{-1} + \mathcal{S}^{\frac{1}{2}}\mathcal{L}(\mathcal{C}_r\mathcal{V}{\mathcal{C}_r}^T)^{-\frac{1}{2}}, \\ &\mathcal{S} = \mathcal{R}^{-1} - \mathcal{R}^{-1}\mathcal{B}_l^T[\mathcal{V} - \mathcal{V}\mathcal{C}_r^T(\mathcal{C}_r\mathcal{V}{\mathcal{C}_r}^T)^{-1}\mathcal{C}_r\mathcal{V}]\mathcal{B}_l\mathcal{R}^{-1}, \\ &\mathcal{V} = (\mathcal{B}_r\mathcal{R}^{-1}{\mathcal{B}_l}^T - \mathcal{D})^{-1} > 0, \end{aligned}$$

where $\mathcal{Z}$, $\mathcal{L}$, $\mathcal{R}$ are arbitrary matrices such that $||\mathcal{L}|| < 1$ and $\mathcal{R} > 0$. Let $A^{\perp}$ be the orthogonal of the matrix A and A^+ be any generalized inverse of the matrix A, verifying $AA^+A = A$.

Lemma 3. *(Xu, 2002) Let $\mathcal{M}$ and $\mathcal{N}$ be two constant matrices of appropriate dimensions. Then, the following inequality:*

$$\mathcal{M}^T\mathcal{N} + \mathcal{N}^T\mathcal{M} \leq \gamma\mathcal{M}^T\mathcal{M} + \frac{1}{\gamma}\mathcal{N}^T\mathcal{N}$$

holds for any scalar $\gamma > 0$.

Chapter 4

Generalized dynamic methodology for algebro-differential-NLPV systems

4.1 Introduction

This chapter addresses with the state estimation problem for nonlinear algebro-differential parameter-varying systems (S-NLPV). The design of a generalized observer for NLPV systems whith a general parameterization is shown. The observer design it is formulated as a set of LMIs. Finally, to demonstrate the effectiveness of the observer, the obtained results are applied to a rolling disc connected to a fixed wall by a spring and a damper.

4.2 Generalized dynamic observer design for S-NLPV systems

Consider the following nonlinear algebro-differential parameter-varying system in its polytopic form.

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[A_i x(t) + B_i u(t) + D_i f(t, F_L x)\Big], \\ y(t) &= Cx(t), \end{aligned} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the known input and $y(t) \in \mathbb{R}^p$ is the measurement output vector, $\varrho(t) \in \mathbb{R}^r$ is the vector of r variable parameters, $\mu_i(\varrho(t))$ are the weighting functions that depend on the variation of $\varrho(t)$. Matrix $E \in \mathbb{R}^{n\times n}$ could be singular. $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D_i \in \mathbb{R}^{n\times n_f}$ are real matrices and $f(t, F_L x) \in \mathbb{R}^{n_f}$ represents a nonlinearity that satisfies the Lipschitz constrain $||\Delta_f|| \leq \lambda ||F_L(x_1 - x_2)||$ where $\Delta_f = f(t, F_L x_1) - f(t, F_L x_2)$, λ represents a known Lipschitz constant, while F_L is a real matrix of suitable dimensions

Let $rank(E) = r < n$ and let $E^{\perp} \in \mathbb{R}^{s\times n}$ be a full row matrix such that $E^{\perp}E = 0$, in this case $s = n - r$.

Considering $\mu_i(\varrho(t))$ as membership functions constructed with predefined variant parameters$\varrho(t) \in \mathbb{R}^l$. The membership functions have the following properties:

$$\sum_{i=1}^{k} \mu_i(\varrho(t)) = 1, \quad \mu_i(\varrho(t)) \geq 0, \tag{4.2}$$

for i ranging from 1 to k, where $k = 2^l$, representing the number of vertices in the polytope

The following definitions and theorem will be referenced throughout the rest of this chapter.

Assumption 1. *It is assumed that system* (4.1) *is regular (Definition 1), Impulse observable (Definition 2) and Reachable observable (Definition 4).*

Assumption 2. *The Lemma 1, elimination Lemma (lemma 2) and Lemma 3 are used in the sequel of the chapter.*

4.2.1 Problem Statement.

Let us consider the following generalized nonlinear observer for system (4.1)

$$\dot{\zeta}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[N_i\zeta(t) + H_iv(t) + F_iy(t) + TD_if(t, F_L\hat{x}) + J_iu(t)\Big], \tag{4.3}$$

$$\dot{v}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[S_i\zeta(t) + L_iv(t) + M_iy(t)\Big], \tag{4.4}$$

$$\hat{x}(t) = P\zeta(t) + Qy(t), \tag{4.5}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector and $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$. The matrices $N_i \in \mathbb{R}^{q_0 \times q_0}$, $H_i \in \mathbb{R}^{q_0 \times q_1}$, $F_i \in \mathbb{R}^{q_0 \times p}$, $S_i \in \mathbb{R}^{q_1 \times q_0}$, $L_i \in \mathbb{R}^{q_1 \times q_1}$, $M_i \in \mathbb{R}^{q_1 \times p}$, $J_i \in \mathbb{R}^{q_0 \times m}$, $P \in \mathbb{R}^{n \times q_0}$, $Q \in \mathbb{R}^{n \times p}$, and $T \in \mathbb{R}^{q_0 \times n}$ are unknown matrices of suitable dimensions that need to be determined to ensure asymptotic convergence of $\hat{x}(t)$ to $x(t)$.
Consider a matrix parameter T to define the following transformed error

$$\epsilon(t) = \zeta(t) - TEx(t), \tag{4.6}$$

its derivative is given as

$$\begin{aligned}
\dot{\epsilon}(t) &= \dot{\zeta}(t) - TE\dot{x}(t)\\
\dot{\epsilon}(t) &= \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[N_i\epsilon(t) + N_iTEx(t) + H_iv(t) + F_iCx(t) + TD_if(t, F_L\hat{x}) +\\
&\quad Ju(t) - TA_ix(t) - TB_iu(t) - TD_if(t, F_Lx)\Big] \qquad (4.7)\\
&= \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[N_i\epsilon(t) + (N_iTE - TA_i + F_iC)x(t) + (J_i - TB_i)u(t) + TD_i\Delta_f(t) + H_iv(t)\Big] \qquad (4.8)
\end{aligned}$$

where $\Delta_f(t) = f(t, F_L\hat{x}) - f(t, F_L x)$. Equation (4.8) is independent of $x(t)$ and $u(t)$, if the following equations are satisfied:

$$(a)\;\; N_iTE + F_iC - TA_i = 0,$$
$$(b)\;\; J_i = TB_i.$$

Then equation (4.8) becomes:

$$\dot{\epsilon}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[N_i\epsilon(t) + H_i v(t) + TD_i\Delta_f(t)\Big]. \tag{4.9}$$

By using equation (4.6), equations (4.4) and (4.5) can be written as

$$\dot{v}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[S_i\epsilon(t) + (S_iTE + M_iC)x(t) + L_i v(t)\Big], \tag{4.10}$$
$$\hat{x}(t) = P\epsilon(t) + (PTE + QC)x(t), \tag{4.11}$$

Now if the following conditions are satisfied

$$(c)\;\; S_iTE + M_iC = 0,$$
$$(d)\;\; PTE + QC = I_n,$$

Convergence is ensured if the aforementioned restrictions (a) - (d) are satisfied. However, if these restrictions are not met, the convergence of the observer cannot be guaranteed.

then equation (4.10) becomes

$$\dot{v}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t)\Big[S_i\epsilon(t) + L_i v(t)\Big], \tag{4.12}$$

and the state estimation error becomes

$$\hat{x}(t) - x(t) = e(t) = P\epsilon(t). \tag{4.13}$$

If conditions *(a)–(d)* are satisfied, the following observer error dynamics equation is obtained from (4.9) and (4.12)

$$\dot{\varphi}(t) \;\;=\;\; \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[\mathbb{A}_i\varphi(t) + \mathbb{B}_i\Delta_f(t)\Big], \tag{4.14}$$

where $\varphi(t) = \begin{bmatrix} \epsilon(t) \\ v(t) \end{bmatrix}$, $\mathbb{A}_i = \begin{bmatrix} N_i & H_i \\ S_i & L_i \end{bmatrix}$ and $\mathbb{B}_i = \begin{bmatrix} TD_i \\ 0 \end{bmatrix}$.

To study the stability of system (4.14) when $\Delta_f(t) = 0$, let us consider $V(t) = \varphi(t)^T X \varphi(t)$ with $X = X^T > 0$ be a Lyapunov function candidate. Then we have

$$\dot{V}(t) = \varphi^T(t)[\mathbb{A}(\varrho)^T X + X\mathbb{A}(\varrho)]\varphi(t)$$

where $\mathbb{A}(\varrho) = \sum_{i=1}^{k} \mu_i(\varrho(t))\mathbb{A}_i$. In this case $\dot{V}(t) < 0$ if $\mathbb{A}(\varrho)^T X + X\mathbb{A}(\varrho) < 0$ which equivalent to

$$\sum_{i=1}^{k} \mu_i(\varrho(t))[\mathbb{A}_i^T X + X\mathbb{A}_i] < 0$$

since $\mu_i \geq 0$ and $\sum_{i=1}^{k} \mu_i(\varrho(t)) = 1$, the stability condition reduces to the one of matrices $\mathbb{A}_i = \begin{bmatrix} N_i & H_i \\ S_i & L_i \end{bmatrix}$.

4.2.2 Observer parameterization

The parameterization can be obtained by first examining constraints *(c)* and *(d)*, which can be formulated as

$$\begin{bmatrix} S_i & M_i \\ P & Q \end{bmatrix} \begin{bmatrix} TE \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ I_n \end{bmatrix}, \tag{4.15}$$

the necessary and sufficient condition for (4.15) to have a solution is

$$rank \begin{bmatrix} TE \\ C \\ I_n \end{bmatrix} = rank \begin{bmatrix} TE \\ C \end{bmatrix} = n. \tag{4.16}$$

From equation (4.16) there always exist two matrices $T \in \mathbb{R}^{q_0 \times n}$ and $K \in \mathbb{R}^{q_0 \times p}$ such that

$$TE + KC = R, \tag{4.17}$$

where $R \in \mathbb{R}^{q_0 \times n}$ is an arbitrary full row rank matrix such that $rank \begin{bmatrix} R \\ C \end{bmatrix} = n$.
Equation (4.17) we can also written as

$$\begin{bmatrix} T & K \end{bmatrix} \underbrace{\begin{bmatrix} E \\ C \end{bmatrix}}_{\Omega} = R, \tag{4.18}$$

and since $rank \begin{bmatrix} \Omega \\ R \end{bmatrix} = rank(\Omega)$, the solution of (4.18) is given by

$$\begin{bmatrix} T & K \end{bmatrix} = R\Omega^+ - Z_1(I_{n+p} - \Omega\Omega^+), \tag{4.19}$$

where Ω^+ denotes the generalized inverse matrix such that $\Omega\Omega^+\Omega = \Omega$.
Therefore

$$T = \underbrace{R\Omega^+ \begin{bmatrix} I_n \\ 0 \end{bmatrix}}_{T_1} - Z_1 \underbrace{(I_{n+p} - \Omega\Omega^+) \begin{bmatrix} I_n \\ 0 \end{bmatrix}}_{T_2}, \tag{4.20}$$

$$K = \underbrace{R\Omega^+ \begin{bmatrix} 0 \\ I_p \end{bmatrix}}_{K_1} - Z_1 \underbrace{(I_{n+p} - \Omega\Omega^+) \begin{bmatrix} 0 \\ I_p \end{bmatrix}}_{K_2}, \tag{4.21}$$

where Z_1 is an arbitrary matrix. By replacing $TE = R - KC$ into condition *(a)*, the following equation is obtained

$$N_iR + \underbrace{(F_i - N_iK)}_{\bar{K}_i} C = TA_i, \tag{4.22}$$

$$\begin{bmatrix} N_i & \bar{K}_i \end{bmatrix} \underbrace{\begin{bmatrix} R \\ C \end{bmatrix}}_{\Sigma} = TA_i, \tag{4.23}$$

The general solution to the equation (4.23) is as follows:

$$\begin{bmatrix} N_i & \bar{K}_i \end{bmatrix} = TA_i\Sigma^+ - Y_{1i}(I_{q_o+p} - \Sigma\Sigma^+), \tag{4.24}$$

by replacing T from (4.20) into (4.24) we have

$$N_i = N_{1i} - Z_1N_{2i} - Y_{1i}N_3, \tag{4.25}$$

$$\bar{K}_i = \bar{K}_{1i} - Z_1\bar{K}_{2i} - Y_{1i}\bar{K}_3, \tag{4.26}$$

where $N_{1i} = T_1A_i\Sigma^+ \begin{bmatrix} I_{q_0} \\ 0 \end{bmatrix}$, $N_{2i} = T_2A_i\Sigma^+ \begin{bmatrix} I_{q_0} \\ 0 \end{bmatrix}$, $N_3 = (I_{q_0+p} - \Sigma\Sigma^+) \begin{bmatrix} I_{q_0} \\ 0 \end{bmatrix}$,
$\bar{K}_{1i} = T_1A_i\Sigma^+ \begin{bmatrix} 0 \\ I_p \end{bmatrix}$, $\bar{K}_{2i} = T_2A_i\Sigma^+ \begin{bmatrix} 0 \\ I_p \end{bmatrix}$, $\bar{K}_3 = (I - \Sigma\Sigma^+) \begin{bmatrix} 0 \\ I_p \end{bmatrix}$ and Y_{1i} is an arbitrary matrix with the necessary dimensions.
Now, from (4.22) we can deduce the value of F_i as

$$\begin{aligned} F_i &= \bar{K}_i + N_iK, \\ &= \bar{K}_{1i} - Z_1\bar{K}_{2i} - Y_{1i}\bar{K}_3 + N_{1i}K - Z_1N_{2i}K - Y_{1i}N_3K, \\ &= F_{1i} - Z_1F_{2i} - Y_{1i}F_3, \end{aligned} \tag{4.27}$$

where $F_{1i} = T_1A_i\Sigma^+ \begin{bmatrix} K \\ I_p \end{bmatrix}$, $F_{2i} = T_2A_i\Sigma^+ \begin{bmatrix} K \\ I_p \end{bmatrix}$ and $F_3 = (I_{q_0+p} - \Sigma\Sigma^+) \begin{bmatrix} K \\ I_p \end{bmatrix}$.
On the other hand, equation (4.17) can be written as

$$\begin{bmatrix} TE \\ C \end{bmatrix} = \begin{bmatrix} I_{q_0} & -K \\ 0 & I_p \end{bmatrix} \Sigma, \tag{4.28}$$

by using (4.15) and (4.28) we obtain

$$\begin{bmatrix} S_i & M_i \\ P & Q \end{bmatrix} \begin{bmatrix} I_{q_0} & -K \\ 0 & I_p, \end{bmatrix} \Sigma = \begin{bmatrix} 0 \\ I_n \end{bmatrix}, \tag{4.29}$$

which leads the following solution

$$\begin{bmatrix} S_i & M_i \\ P & Q \end{bmatrix} = \left(\begin{bmatrix} 0 \\ I_n \end{bmatrix} \Sigma^+ - \begin{bmatrix} Y_{2i} \\ Y_{3i} \end{bmatrix} (I_{q_0+p} - \Sigma\Sigma^+) \right) \begin{bmatrix} I_{q_0} & K \\ 0 & I_p \end{bmatrix}, \tag{4.30}$$

where Y_{2i} and Y_{3i} are arbitrary matrices of appropriate dimensions, also we have used the fact that $\begin{bmatrix} I_{q_0} & -K \\ 0 & I_p \end{bmatrix}^{-1} = \begin{bmatrix} I_{q_0} & K \\ 0 & I_p \end{bmatrix}$.

From (4.30) it can be deduced the general form of matrices S_i, M_i, P and Q as follows

$$S_i = -Y_{2i}N_3, \tag{4.31}$$
$$M_i = -Y_{2i}F_3, \tag{4.32}$$
$$P = P_1 - Y_{3i}N_3, \tag{4.33}$$
$$Q = Q_1 - Y_{3i}F_3, \tag{4.34}$$

where $F_3 = (I_{q_0+p} - \Sigma\Sigma^+)\begin{bmatrix} K \\ I_p \end{bmatrix}$, $N_3 = (I_{q_0+p} - \Sigma\Sigma^+)\begin{bmatrix} I_{q_0} \\ 0 \end{bmatrix}$, $Q_1 = \Sigma^+\begin{bmatrix} K \\ I_p \end{bmatrix}$ and $P_1 = \Sigma^+\begin{bmatrix} I_{q_0} \\ 0 \end{bmatrix}$.

Now, by using the value of matrices N_i, S_i and T given by (4.25), (4.31), (4.20), the observer error dynamics (4.14) can be written as

$$\dot{\varphi}(t) \;=\; \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[(\mathbb{A}_{1i} - \mathbb{Y}_i\mathbb{A}_2)\varphi(t) + \mathbb{B}_i\Delta_f(t)\Big], \tag{4.35}$$

where $\mathbb{A}_{1i} = \begin{bmatrix} N_{1i} - Z_1N_{2i} & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{B}_i = \begin{bmatrix} (T_1 - Z_1T_2)D_i \\ 0 \end{bmatrix}$ and $\mathbb{Y}_i = \begin{bmatrix} Y_{1i} & H_i \\ Y_{2i} & L_i, \end{bmatrix}$, and from (4.13) we have

$$e(t) = \mathbb{P}\varphi(t), \tag{4.36}$$

where $\mathbb{P} = \begin{bmatrix} P_1 & 0 \end{bmatrix}$, without loss of generality $Y_{3i} = 0$ is taken for simplicity.
The problem of the observer design is reduced to determine matrices Z_1 and $\mathbb{Y}_i$ such that system (4.35) is asymptotically stable.

4.2.3 Stability analysis of the observer

This section is devoted to the stability analysis of equation (4.35). The following theorem gives the condition for the stability in a set of LMIs.

Theorem 1. *Consider the system given by* (4.3)-(4.5) *as a generalized observer for system* (4.35) *for state estimation if Assumption 1 is fulfilled. The observer* (4.3)-(4.5) *is asymptotically stable if there exists two parameter matrices* Z_1, $\mathbb{Y}_i$ *and a symmetric matrix* $X_1 = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$ *with* $X_{11} = \begin{bmatrix} X_a & X_a \\ X_a & X_b \end{bmatrix}$ *such that the following LMIs are satisfied*

$$\mathcal{C}^{T\perp}\begin{bmatrix} \mathcal{H}_e(\Pi_{1i}^T + \mathbb{P}X_{12}^T + \mathbb{P}^TU_1) & (*) & (*) \\ U_2^T\mathbb{P} - X_{12}^T - U_1 & -U_2 - U_2^T + \gamma\lambda^2F_LF_L^T & 0 \\ \Pi_{2i}^T & 0 & -\gamma I_{n+q_0+q_1} \end{bmatrix}\mathcal{C}^{T\perp T} < 0, \tag{4.37}$$

$$\begin{bmatrix} -U_2 - U_2^T + \gamma\lambda^2F_L^TF_L & 0 \\ 0 & -\gamma I_{n+q_0+q_1} \end{bmatrix} < 0, \tag{4.38}$$

where

$$\mathcal{C}^{T\perp} = \begin{bmatrix} \mathbb{A}_2^{T\perp} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \Pi_{1i} = \begin{bmatrix} X_a N_{1i} - X_z N_{2i} & 0 \\ X_a N_{1i} - X_z N_{2i} & 0 \end{bmatrix} \textit{ and } \Pi_{2i} = \begin{bmatrix} X_a T_1 D_i - X_z T_2 D_i \\ X_a T_1 D_i - X_z T_2 D_i \end{bmatrix}, \tag{4.39}$$

$\mathcal{H}_e$ is known as hermitian equation, in this case matrix $Z_1 = X_a^{-1} X_z$ and parameter matrix Φ_i is obtained as follows

$$\Phi_i = \mathcal{B}_r{}^+ \mathcal{K}_i \mathcal{C}_l{}^+ + \mathcal{Z} - \mathcal{B}_r{}^+ \mathcal{B}_r \mathcal{Z} \mathcal{C}_l \mathcal{C}_l{}^+, \tag{4.40}$$

where

$$\mathbb{Y}_i = \Phi_i X_{11}^{-1}, \tag{4.41}$$

$$\mathcal{K}_i = -\mathcal{R}^{-1} \mathcal{B}_l{}^T \mathcal{V}_i \mathcal{C}_r{}^T (\mathcal{C}_r \mathcal{V}_i \mathcal{C}_r{}^T)^{-1} + \mathcal{S}_i^{\frac{1}{2}} \mathcal{L} (\mathcal{C}_r \mathcal{V}_i \mathcal{C}_r{}^T)^{-\frac{1}{2}}, \tag{4.42}$$

$$\mathcal{S}_i = \mathcal{R}^{-1} - \mathcal{R}^{-1} \mathcal{B}_l^T [\mathcal{V}_i - \mathcal{V}_i \mathcal{C}_r^T (\mathcal{C}_r \mathcal{V}_i \mathcal{C}_r{}^T)^{-1} \mathcal{C}_r \mathcal{V}_i] \mathcal{B}_l \mathcal{R}^{-1}, \tag{4.43}$$

$$\mathcal{V}_i = (\mathcal{B}_r \mathcal{R}^{-1} \mathcal{B}_l{}^T - \mathcal{D}_i)^{-1} > 0, \tag{4.44}$$

with matrices $\mathcal{Z}$, $\mathcal{L}$ and $\mathcal{R}$ as arbitrary matrices of appropriate dimensions satisfying $||\mathcal{L}|| < 1$ and $\mathcal{R} > 0$, with

$$\mathcal{D}_i = \begin{bmatrix} \mathcal{H}_e(\Pi_{1i}^T + \mathbb{P} X_{12}^T + \mathbb{P}^T U_1) & (*) & (*) \\ U_2^T \mathbb{P} - X_{12}^T - U_1 & -U_2 - U_2^T + \gamma \lambda^2 F_L F_L^T & 0 \\ \Pi_{2i}^T & 0 & -\gamma I_{n+q_0+q_1} \end{bmatrix},$$

$\mathcal{B} = \begin{bmatrix} -I \\ 0 \\ 0 \end{bmatrix}$ *and* $\mathcal{C} = \begin{bmatrix} \mathbb{A}_2 & 0 & 0 \end{bmatrix}$. *Matrices $\mathcal{B}_l$, $\mathcal{B}_r$, $\mathcal{C}_l$ and $\mathcal{C}_r$ are any full rank matrices such that $\mathcal{B} = \mathcal{B}_l \mathcal{B}_r$ and $\mathcal{C} = \mathcal{C}_l \mathcal{C}_r$.*

Proof. Equation (4.35) can be rewritten in singular form as

$$\bar{\mathbb{E}} \dot{\xi}(t) \;=\; \sum_{i=1}^{k} \mu_i(\varrho(t)) \Big[\bar{\mathbb{A}}_i \xi(t) + \bar{\mathbb{B}}_i \Delta_f(t) \Big], \tag{4.45}$$

where $\xi(t) = \begin{bmatrix} \varphi(t) \\ e(t) \end{bmatrix}$, $\bar{\mathbb{E}} = \begin{bmatrix} I_{q_0+q_1} & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{\mathbb{A}}_i = \begin{bmatrix} \mathbb{A}_{1i} - \mathbb{Y}_i \mathbb{A}_2 & 0 \\ \mathbb{P} & -I_n \end{bmatrix}$, $\bar{\mathbb{B}}_i = \begin{bmatrix} \mathbb{B}_i \\ 0 \end{bmatrix}$.
Now, by considering the following Lyapunov function (Wu et al., 2013)

$$V(\xi(t)) = \xi(t)^T \bar{\mathbb{E}}^T X \xi(t), \tag{4.46}$$

where

$$\bar{\mathbb{E}}^T X = X^T \bar{\mathbb{E}} \geq 0, \tag{4.47}$$

and $X = X_1 \bar{\mathbb{E}} + \bar{\mathbb{E}}^{\perp T} U$, with matrices $X_1 = X_1^T = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} > 0$ and $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$, such that from (4.47) we can obtain

$$\bar{\mathbb{E}}^T X_1 \bar{\mathbb{E}} = \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \tag{4.48}$$

with $X_{11} = X_{11}^T = \begin{bmatrix} X_a & X_a \\ X_a & X_b \end{bmatrix} > 0$.
The derivative of (4.46) along the trajectory of (4.45) gives

$$\dot{V}(\xi(t)) = \sum_{i=1}^{k} \mu_i(\varrho(t)) \Big[\xi^T(t)(\bar{\mathbb{A}}_i^T X + X^T \bar{\mathbb{A}}_i)\xi(t) + \boxed{\Delta_f(t)^T \bar{\mathbb{B}}_i^T X \xi(t) + \xi^T(t) X^T \bar{\mathbb{B}}_i \Delta_f(t)} \Big], \quad (4.49)$$

By using Lemma 3 we can obtain the following inequality from the framed section in equation (4.49)

$$\Delta_f^T(t)\bar{\mathbb{B}}_i^T X\xi(t) + \xi^T(t)X^T\bar{\mathbb{B}}_i\Delta_f(t) \leq \boxed{\gamma\Delta_f^T(t)\Delta_f(t)} + \frac{1}{\gamma}\xi^T(t)X^T\bar{\mathbb{B}}_i\bar{\mathbb{B}}_i^T X\xi(t), \quad (4.50)$$

using the Lipschitz condition in the framed section of equation (4.50), we have

$$\Delta_f^T(t)\Delta_f(t) \leq \xi^T(t) \begin{bmatrix} 0 & 0 \\ 0 & \lambda^2 F_L^T F_L, \end{bmatrix} \xi(t), \quad (4.51)$$

by inserting (4.50) and (4.51) into (4.49), we obtain

$$\dot{V}(\xi(t)) \leq \sum_{i=1}^{k} \mu_i(\varrho(t)) \left[\xi^T(t) \left(\bar{\mathbb{A}}_i^T X + X^T \bar{\mathbb{A}}_i + \frac{1}{\gamma} X^T \bar{\mathbb{B}}_i \bar{\mathbb{B}}_i^T X + \gamma \begin{bmatrix} 0 & 0 \\ 0 & \lambda^2 F_L^T F_L \end{bmatrix} \right) \xi(t) \right]. \quad (4.52)$$

Now, if the following LMI is satisfied then $\dot{V}(\xi(t)) < 0$.

$$\bar{\mathbb{A}}_i^T X + X^T \bar{\mathbb{A}}_i + \frac{1}{\gamma} X^T \bar{\mathbb{B}}_i \bar{\mathbb{B}}_i^T X + \gamma \begin{bmatrix} 0 & 0 \\ 0 & \lambda^2 F_L^T F_L \end{bmatrix} < 0, \quad (4.53)$$

By using the Schur complement (Lemma 1) we obtain

$$\begin{bmatrix} \bar{\mathbb{A}}_i^T X + X^T \bar{\mathbb{A}}_i + \gamma \begin{bmatrix} 0 & 0 \\ 0 & \lambda^2 F_L^T F_L \end{bmatrix} & (*) \\ \bar{\mathbb{B}}_i^T X & -\gamma I_{n+q_0+q_1} \end{bmatrix} < 0. \quad (4.54)$$

By replacing matrices $\bar{\mathbb{A}}_i$, $\bar{\mathbb{B}}_i$, and X in inequality (4.54), we obtain

$$\begin{bmatrix} \mathcal{H}_e(\mathbb{A}_{1i}^T X_{11} - \mathbb{A}_2^T \Phi_i^T + \mathbb{P} X_{12}^T + \mathbb{P}^T U_1) & (*) & (*) \\ U_2^T \mathbb{P} - X_{12}^T - U_1 & -U_2 - U_2^T + \gamma\lambda^2 F_L^T F_L & 0 \\ \mathbb{B}_i^T X_{11} & 0 & -\gamma I_{n+q_0+q_1} \end{bmatrix} < 0, \quad (4.55)$$

where $\Phi_i = X_{11}\mathbb{Y}_i$.
Now, by inserting the values of matrices $\mathbb{A}_{1i}$, $\mathbb{B}_i$ and X_{11} we obtain

$$\begin{bmatrix} \mathcal{H}_e(\Pi_{1i}^T - \mathbb{A}_2^T \Phi_i^T + \mathbb{P} X_{12}^T + \mathbb{P}^T U_1) & (*) & (*) \\ U_2^T \mathbb{P} - X_{12}^T - U_1 & -U_2 - U_2^T + \gamma\lambda^2 F_L^T F_L & 0 \\ \Pi_{2i}^T & 0 & -\gamma I_{n+q_0+q_1} \end{bmatrix} < 0, \quad (4.56)$$

where Π_{1i} and Π_{2i} are defined in (4.39), $X_z = X_a Z_1$.
Inequality (4.56) can also be rewritten as

$$(\mathcal{B}\Phi_i\mathcal{C}) + (\mathcal{B}\Phi_i\mathcal{C})^T + \mathcal{D}_i < 0, \tag{4.57}$$

where $\mathcal{D}_i = \begin{bmatrix} \mathcal{H}_e(\Pi_{1i}^T + \mathbb{P}X_{12}^T + \mathbb{P}^T U_1) & (*) & (*) \\ U_2^T\mathbb{P} - X_{12}^T - U_1 & -U_2 - U_2^T + \gamma\lambda^2 F_L^T F_L & 0 \\ \Pi_{2i}^T & 0 & -\gamma I_{n+q_0+q_1} \end{bmatrix}$,

$\mathcal{B} = \begin{bmatrix} -I \\ 0 \\ 0 \end{bmatrix}$ and $\mathcal{C} = \begin{bmatrix} \mathbb{A}_2 & 0 & 0 \end{bmatrix}$.

According to Lemma 2, inequality (4.57) is satisfied if and only if the following inequalities verified

$$\mathcal{C}^{T\perp}\mathcal{D}_i\mathcal{C}^{T\perp T} < 0, \tag{4.58}$$

$$\mathcal{B}^{\perp}\mathcal{D}_i\mathcal{B}^{\perp T} < 0, \tag{4.59}$$

the inequalities (4.58) and (4.59) are equivalent to (4.60) and (4.61) respectively

$$\mathcal{C}^{T\perp} \begin{bmatrix} \mathcal{H}_e(\Pi_{1i}^T + \mathbb{P}X_{12}^T + \mathbb{P}^T U_1) & (*) & (*) \\ U_2^T\mathbb{P} - X_{12}^T - U_1 & -U_2 - U_2^T + \gamma\lambda^2 F_L^T F_L & 0 \\ \Pi_{2i}^T & 0 & -\gamma I_{n+q_0+q_1} \end{bmatrix} \mathcal{C}^{T\perp T} < 0, \tag{4.60}$$

$$\begin{bmatrix} -U_2 - U_2^T + \gamma\lambda^2 F_L^T F_L & 0 \\ 0 & -\gamma I_{n+q_0+q_1} \end{bmatrix} < 0. \tag{4.61}$$

which complete the proof of the theorem. □

Remark

The following algorithm summarizes the procedure to design a GDNPVO.

1. Select the observer order q_0 and a matrix $R \in \mathbb{R}^{q_0 \times n}$ such that $rank(\Sigma) = n$.

2. Compute matrices N_{1i}, N_{2i}, N_3, T_1, T_2, K_1, K_2 and P_1 defined in Section 4.2.2.

3. Solve LMIs (4.60) and (4.61) to find X, Z_1 and $\gamma > 0$.

4. Propose matrices $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{Z}$, such that $||\mathcal{L}|| < 1$, $\mathcal{R} > 0$ and $\mathcal{V}_i > 0$.

5. Considering the elimination lemma, determine the parameter matrix $\mathbb{Y}_i$ as in equation (4.41), which involves the unknown observer matrices.

6. Compute all the remaining matrices of the GDNPVO (4.3)-(4.5); N_i, H_i, F_i, J_i, S_i, L_i, M_i, P and Q, by using (4.25) to compute N_i, (4.41) to compute H_i and L_i, (4.31)-(4.34) to compute S_i, M_i, P and Q taking matrix $Y_3 = 0$. The matrices F_i are given by (4.27) and matrix J_i from constraint (b).

4.2.4 Particular cases

This section shows how the proportional observer (PO) and proportional integral-observer (PIO) are particular cases of the generalized observer (GO), and how they can be designed directly from our results.

Proportional observer (PO)

Considering the nonlinear algebro-differential parameter-varying system of the equation (4.1), the PO corresponds to the following considerations in the matrices of the generalized nonlinear observer of equations (4.3)-(4.5): $H_i = 0$, $S_i = 0$, $M_i = 0$ and $L_i = 0$. In this case we obtain the following observer:

$$\dot{\zeta}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[N_i\zeta(t) + F_i y(t) + TD_i f(t, F_L\hat{x}) + J_i u(t)\Big], \tag{4.62}$$

$$\hat{x}(t) = P\zeta(t) + Qy(t), \tag{4.63}$$

and the observer error dynamics (4.35) becomes:

$$\dot{\epsilon}(t) \;\; = \;\; \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[(\mathbb{A}_{1i} - \mathbb{Y}_i\mathbb{A}_2)\epsilon(t) + \mathbb{B}_i\Delta f(t)\Big], \tag{4.64}$$

where $\mathbb{A}_{1i} = N_{1i} - Z_1N_{2i}$, $\mathbb{A}_2 = N_3$, $\mathbb{B}_i = (T_1 - Z_1T_2)D_i$ and $\mathbb{Y}_i = Y_{1i}$.

With these matrices the observer design can be directly obtained following the results of Section 4.2.2 and 4.2.3.

Proportional-integral observer (PIO)

The PIO corresponds to the following considerations in the matrices of the generalized nonlinear observer of equations (4.3)-(4.5): $L_i = 0$, $S_i = CP$ and $M_i = Q - I$. Obtaining the following observer:

$$\dot{\zeta}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[N_i\zeta(t) + H_i v(t) + F_i y(t) + TD_i f(t, F_L\hat{x}) + J_i u(t)\Big], \tag{4.65}$$

$$\dot{v}(t) = C\hat{x}(t) - y(t), \tag{4.66}$$

$$\hat{x}(t) = P\zeta(t) + Qy(t), \tag{4.67}$$

and the observer error dynamics (4.35) becomes:

$$\dot{\varphi}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[(\mathbb{A}_{1i} - \mathbb{Y}_i\mathbb{A}_2)\varphi(t) + \mathbb{B}_i\Delta f(t)\Big], \tag{4.68}$$

where $\mathbb{A}_{1i} = \begin{bmatrix} N_{1i} - Z_1N_{2i} & 0 \\ -CP_1 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I \end{bmatrix}$, $\mathbb{B}_i = \begin{bmatrix} (T_1 - Z_1T_2)D_i \\ 0 \end{bmatrix}$ and $\mathbb{Y}_i = \begin{bmatrix} I \\ 0 \end{bmatrix}\begin{bmatrix} Y_{1i} & H_i \end{bmatrix}$.

The design can be obtained directly by applying the results of section 4.2.2 and 4.2.3.

4.3 Application to a rolling disc

A rolling disc connected to a fixed wall is a mechanical system where a disc can roll on a surface while being attached to a wall via a rod or a spring.

In real-world engineering, the concepts of rolling motion and friction are applied in the design of vehicles and robots. These principles are essential for understanding the movement of wheels and ensuring the stability and efficiency of the systems. Additionally, rolling disc systems are used in manufacturing and transportation, such as in conveyor belts and industrial machinery, to move products efficiently and with minimal energy loss. The design of a state observer for a rolling disc is crucial for enhancing the monitoring and diagnosis of the system through precise estimations of internal variables, as it is often not straightforward due to the complexity and cost of sensors required to directly measure these variables (Rizal et al., 2022).

4.3.1 Model of the Rolling disc

To show the effectiveness of the observer, the obtained results are applied to a rolling disc presented in Sjoberg and Glad (2006). The rolling disc is presented in Figure 4.1, it is connected to a fixed wall by a spring and a damper.

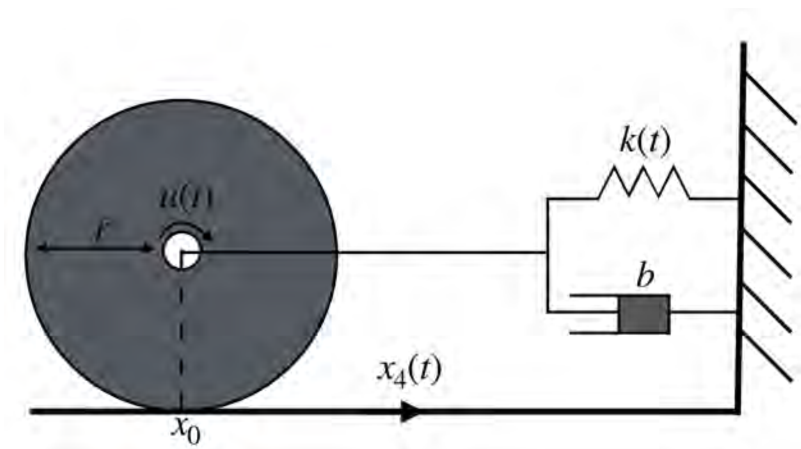

Fig. 4.1. Rolling disc connected to a fixed wall.

This system is governed by the following set of ordinary differential equations

$$\dot{x}_1(t) = x_2(t), \tag{4.69}$$

$$\dot{x}_2(t) = -\frac{k(t)}{m}x_1(t) - \frac{k(t)}{m}x_1^3(t) - \frac{b}{m}x_2(t) - \frac{1}{m}x_4(t), \tag{4.70}$$

$$0 = x_2(t) - rx_3(t), \tag{4.71}$$

$$0 = -\frac{k(t)}{m}x_1(t) - \frac{k(t)}{m}x_1^3(t) - \frac{b}{m}x_2(t) + \Big(\frac{r^2}{J} + \frac{1}{m}\Big)x_4(t) - \frac{r}{J}u(t), \tag{4.72}$$

where $x_1(t)$ is the position of the center of the disc, $x_2(t)$ is the translational velocity of this center, $x_3(t)$ is the angular velocity of the disc, $x_4(t)$ is the contact force between the disc and the surface, $u(t)$ is the applied input force to the disc.

The spring has the positive stiffness $k(t)$. The parameter $b > 0$ is the damping coefficient of the shock absorber. The radius of the disc is r, its inertia is given by J and the mass of the disc is m.
The parameters of the Rolling disc system are given in the following table (Estrada et al., 2015a).

Table 4.1. Rolling disc system parameters

Description	Symbol	Value
Damping coefficient	b	35
Disc radius	r	0.3
Inertia coefficient	J	$3.2kg \cdot m^2$
Mass	m	$40kg$
Stiffness	$k(t)$	$[70,\ 100]N/m$

4.3.2 Nonlinear algebro-differential parameter-varying system formulation

From (4.71) we get $x_2(t) = rx_3(t)$ and replacing it in (4.69)-(4.72), we can obtain the following descriptor model

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} 0 & r & 0 \\ -\frac{\rho(t)}{mr} & -\frac{b}{m} & -\frac{1}{mr} \\ -\frac{\rho(t)}{m} & -\frac{br}{m} & \left(\frac{r^2}{J} + \frac{1}{m}\right) \end{bmatrix}}_{A(\rho(t))} \underbrace{\begin{bmatrix} x_1(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ -\frac{r}{J} \end{bmatrix}}_{B} u(t) + \underbrace{\begin{bmatrix} 0 \\ -\frac{\rho(t)}{mr} \\ -\frac{\rho(t)}{m} \end{bmatrix}}_{D(\rho(t))} \underbrace{x_1^3(t)}_{f(t,F_Lx)}, \tag{4.73}$$

$$y(t) = \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{C} x(t), \tag{4.74}$$

where the stiffness $\rho(t) = k(t)$ is the time varying parameter and the nonlinearity $f(t, F_Lx)$ is considered as a Lipschitz function. The input signal is defined as

$$u(t) = \begin{cases} 0.1Nm & 0s \leq t \leq 15s \\ 0.4Nm & \text{otherwise.} \end{cases} \tag{4.75}$$

with this, the Lipschitz constant is selected as $\lambda = 10$. In fact the state $x_1(t)$ is in a bounded set and $x_1^3(t)$ is Lipschitz as shown below.

Determination of the Lipschitz constant

First we recall that if a function $g(x, u, t)$ is continuous in a closed and bounded region D and if $\dfrac{\partial g}{\partial x}$ is continuous in D, then there exist a positive constant λ such that $\|g(x_1, \mu, t) - g(x_2, \mu, t)\| \leq \lambda \|x_1 - x_2\|$. Now, to calculate this constant for our model, we can put the Jacobian matrix of the system (4.73)-(4.74) as shown below

$$\frac{\partial f(x,u,t)}{\partial x(t)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m} - \frac{3k_2 x_1^2(t)}{m} & \frac{-b}{m} & 0 & \frac{1}{m} \\ 0 & 1 & -r & 0 \\ -\frac{k_1}{m} - \frac{3k_2 x_1^2(t)}{m} & 0 & -\frac{b}{m} & \frac{r^2}{J} + \frac{1}{m} \end{bmatrix} \tag{4.76}$$

since in a finite space all the norms are equivalent, the norm 1 is used in (4.76) to obtain the maximum column value, resulting in

$$\begin{aligned} \left\| \frac{\partial f(x,u,t)}{\partial x(t)} \right\|_1 \leq max \Bigg\{ & \left| -\frac{k_1}{m} \right| + \left| -\frac{3k_2 x_1^2(t)}{m} \right| + \left| -\frac{k_1}{m} \right| + \left| -\frac{3k_2 x_1^2(t)}{m} \right|, |1| + \\ & \left| -\frac{b}{m} \right| + |1|, |-r| + \left| -\frac{b}{m} \right|, \left| \frac{1}{m} \right| + \left| \frac{r^2}{J} + \frac{1}{m} \right| \Bigg\} \end{aligned} \tag{4.77}$$

substituting the parameters values of the Table 4.1 in the equation (4.77) we obtain

$$\left\| \frac{\partial f(x,u,t)}{\partial x(t)} \right\|_1 \leq max \left\{ 5 + 15x_1^2(t), 2.75, 1.15, 0.1 \right\} \tag{4.78}$$

Therefore any constant greater than the maximum value of the equation (4.78) can be selected as the Lipschitz constant λ for this particular example.
In practical case, by considering the value of r given in Table 4.1 and the fact that the maximum length of the considered spring is $d = 0.25m$, the maximum value of x_1 is : $max(x_1) = r + d = 0.55m$.
The maximum value of (4.78) is then obtained as :

$$\left\| \frac{\partial f(x,u,t)}{\partial x(t)} \right\|_1 \leq max \left\{ 5 + 15x_1^2(t), 2.75, 1.15, 0.1 \right\} = 9.5375 \leq \lambda \tag{4.79}$$

which justify the choice of $\lambda = 10$.

4.3.3 Simulation results

First we can see Assumption 1 is satisfied, then we can apply the algorithm of section (4.2.3) as follows

1. By fixing $q_0 = 3$, we can choose matrix $R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times 10^3$, such that $rank(\Sigma) = 3$.

2. We obtain the following matrices $N_{11} = \begin{bmatrix} -2.92 & -2.29 & 5.26 \\ -2.92 & -2.92 & 5.26 \\ -2.92 & -2.92 & 4.96 \end{bmatrix}$,

   $T_1 = \begin{bmatrix} 1000 & 1000 & 0 \\ 1000 & 1000 & 0 \\ 0 & 1000 & 0 \end{bmatrix}$, $N_{12} = \begin{bmatrix} -4.17 & -4.17 & 7.76 \\ -4.17 & -4.17 & 7.76 \\ -4.17 & -4.17 & 7.46 \end{bmatrix}$,

   $T_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $N_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.0009 & -0.0009 & 0.0015 \\ 0 & 0 & 0 \end{bmatrix}$,

   $N_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.0013 & -0.0013 & 0.0022 \\ 0 & 0 & 0 \end{bmatrix}$, $K_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $K_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

   $N_3 = \begin{bmatrix} 0.50 & -0.50 & 0 \\ -0.50 & 0.50 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $P_1 = \begin{bmatrix} 0.5 & 0.5 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-3}$.

3. By using the YALMIP toolbox [Lofberg (2004)] we solve LMIs (4.60) and (4.61) to obtain

   $$X = \begin{bmatrix} 0.0014 & -0.0014 & -0.0005 & 0.0014 & -0.0014 & -0.0005 & 0 & 0 & 0 \\ -0.0014 & 0.0014 & 0.0005 & -0.0014 & 0.0014 & 0.0005 & 0 & 0 & 0 \\ -0.0005 & 0.0005 & 0.0014 & -0.0005 & 0.0005 & 0.0014 & 0 & 0 & 0 \\ 0.0014 & -0.0014 & -0.0005 & 2.3369 & -0.0157 & -0.0004 & 0 & 0 & 0 \\ -0.0014 & 0.0014 & 0.0005 & -0.0157 & 2.3369 & 0.0004 & 0 & 0 & 0 \\ 0.0116 & -0.0155 & 0.0040 & 0 & 0 & 0 & 3.9898 & 0 & 0 \\ 0 & 0 & -0.0040 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \times 10^8,$$

   $Z_1 = \begin{bmatrix} 0 & 0 & -1.4987 & 0 \\ 0 & 0 & -1.5021 & 0 \\ 0 & 0 & 0.0345 & 0 \end{bmatrix} \times 10^5$, and $\gamma = 5.8599 \times 10^{-5}$

4. Considering $\mathcal{R} = I_8 \times 996.3$ and $\mathcal{L} = \text{ ones}_{8,4} \times 0.01$ satisfying the following conditions $||\mathcal{L}|| < 1$, and $\mathcal{R} > 0$ such that $\mathcal{V}_i > 0$.

5. Taking matrix $\mathcal{Z} = 0$ we obtain matrix $\mathbb{Y}_i$ as in equation (4.41).

$$\mathbb{Y}_1 = \begin{bmatrix} 1.0068 & -1.0068 & 0 & 0 & -1.0067 & 1.0067 & 0 \\ 1.0068 & -1.0068 & 0 & 0 & -1.0067 & 1.0067 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times 10^4,$$

$$\mathbb{Y}_2 = \begin{bmatrix} 1.1321 & -1.1321 & 0 & 0 & -1.1320 & 1.1320 & 0 \\ 1.1321 & -1.1321 & 0 & 0 & -1.1320 & 1.1320 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times 10^4$$

6. We deduce all the matrices of the GDNPVO and we obtain

$$N_1 = \begin{bmatrix} -1.0202 & 0.9934 & 0.0228 \\ -1.0202 & 0.9933 & 0.0229 \\ 0 & 0 & 0 \end{bmatrix} \times 10^4, \; J = \begin{bmatrix} -1.4051 \\ -1.4082 \\ 0.0324 \end{bmatrix} \times 10^4,$$

$$N_2 = \begin{bmatrix} -1.1512 & 1.1129 & 0.0343 \\ -1.1513 & 1.1129 & 0.0344 \\ 0 & 0 & 0 \end{bmatrix} \times 10^4, \; P = \begin{bmatrix} 0.5 & 0.5 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-3},$$

$$H_1 = \begin{bmatrix} -1.0067 & -1.0067 & 0 \\ -1.0067 & -1.0067 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^4, \; H_2 = \begin{bmatrix} -1.1320 & 1.1320 & 0 \\ -1.1320 & 1.1320 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^4,$$

$$S_1 = \begin{bmatrix} -1.0068 & 1.0068 & 0 \\ -1.0068 & 1.0068 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^4, \; S_2 = \begin{bmatrix} -1.1321 & 1.1321 & 0 \\ -1.1321 & 1.1321 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^4,$$

$$F_1 = F_2 = \begin{bmatrix} 7.8787 \\ 7.8787 \\ -0.2668 \end{bmatrix} \times 10^3, \; L_1 = \begin{bmatrix} 0.4262 & -0.4266 & 0 \\ 0.4265 & -0.4269 & 0 \\ 0.0231 & -0.0231 & -0.0007 \end{bmatrix} \times 10^{-8},$$

$$L_2 = \begin{bmatrix} 0.4799 & -0.4799 & 0 \\ 0.4796 & -0.4796 & 0 \\ -0.0938 & 0.0938 & -0.0007 \end{bmatrix} \times 10^{-8}, \; M_1 = M_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The input was considered as shown in (4.75). The initial condition for the nonlinear system (4.73) - (4.74) are $x(0) = [0.1,\ 0.75,\ 8.6353]^T$, and for the observer (4.3) - (4.5) the initial conditions are $\hat{x}(0) = [2.1,\ 15.75,\ 735]^T$.

The performance of the proposed observer is shown in in Figures 4.2-4.4. To evaluate the observers robustness, parametric uncertainties were considered in the system during all the simulation with amplitude of $\Delta b = 6 \times 10^{-3}$ and $\Delta J = 1 \times 10^{-5}$, so that in simulation we have $\bar{b} = b - \Delta b$ and $\bar{J} = J - \Delta J$. Figures 4.2-4.3 show the estimation of the system states and Figure 4.4 the algebraic system state.

In order to compare the observer performances, the integral of absolute error (IAE) is calculated in Table 4.2. It can be concluded that the GDO has a good state estimation despite parameter

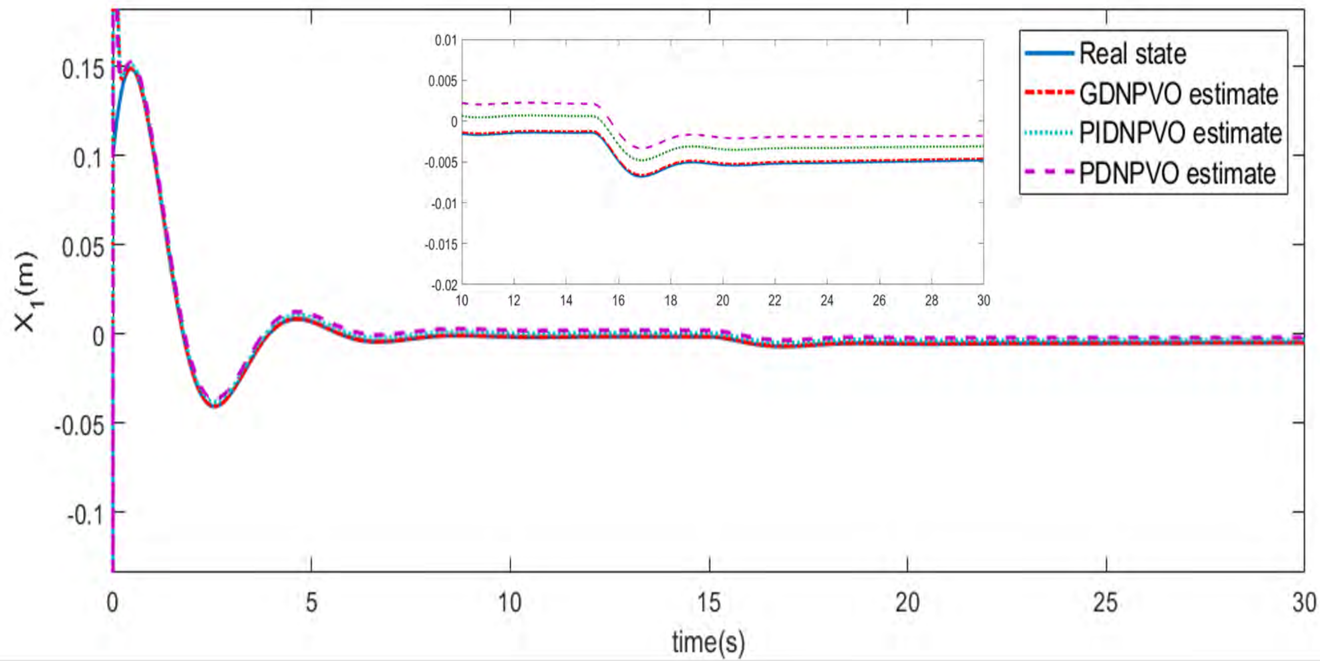

Fig. 4.2. Comparison of state estimate $x_1(t)$.

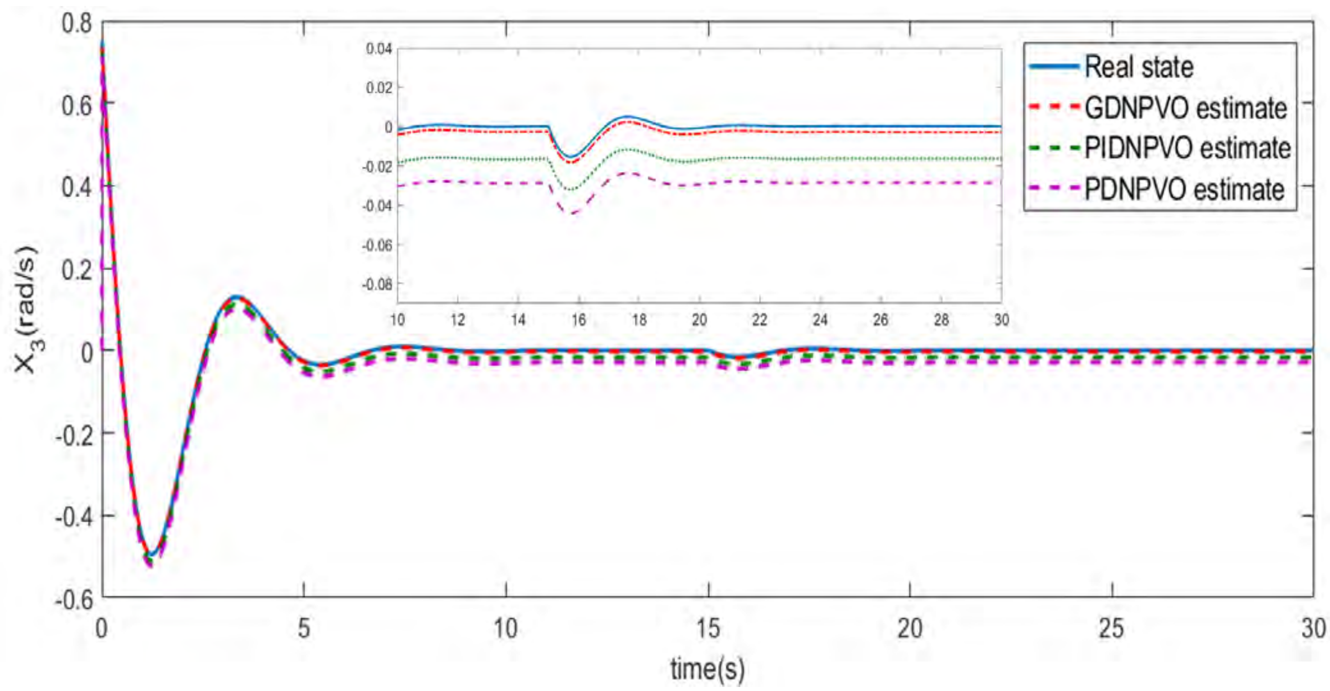

Fig. 4.3. Comparison of state estimate $x_3(t)$.

uncertainties compared with the PO and PIO, obtaining the minimum valued on the estimation

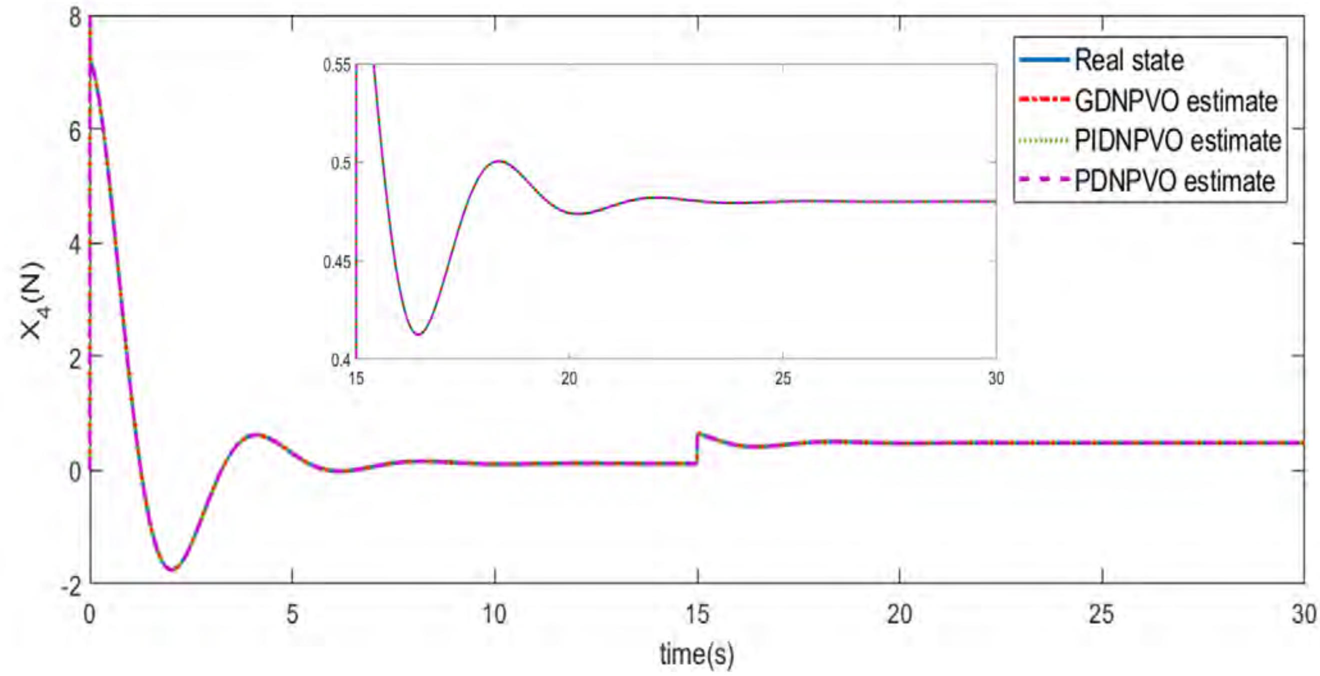

Fig. 4.4. Comparison of state estimate $x_4(t)$.

Table 4.2. Observer performance index

States	**GDO IAE**	**PIO IAE**	**PO IAE**
$\hat{x}_1(t) - x_1(t)$	41.38	70.89	79.91
$\hat{x}_3(t) - x_3(t)$	37.85	249.46	431.62
$\hat{x}_4(t) - x_4(t)$	22.03	22.03	22.03

errors. **The definition of performance analysis can be found in the appendix A**.

4.3.4 Conclusions

In this chapter a GDO for descriptor nonlinear parameter varying systems is synthesized to perform state estimation. The stability conditions is presented through the solution of LMIs. Depending on the design specifications, the GDNPVO can be configured as a simple proportional or a proportional-integral observer by suitably computing the observer gains. A rolling disc model form was considered to demonstrate the performance of the designed observer. It has been demonstrated that the GDNPVO improves robustness in estimation performance against parametric uncertainties.

Chapter 5

Generalized dynamic unknown inputs observers for S-NLPV systems

5.1 Introduction

This chapter presents the simultaneous unknown inputs and state estimation problem for nonlinear algebro-differential parameter-varying systems (DAE-NLPV). This structure is general, and the PO and PIO designs can be considered as particular cases. The performance of the proposed methodology is evaluated in the model of a heat exchanger with two countercurrent cells with actuator faults.

In Section 5.3, the design of a generalized learning observer structure for simultaneous estimation of variable states and actuator faults for the same class of systems, where a general parameterization is shown. This generalized structure provides additional degrees of freedom in the observer design to improve robustness and reduce the convergence time for fault estimation. The design is obtained in terms of a set of linear matrix inequalities evaluated in the model of a heat exchanger.

5.2 Generalized dynamic unknown inputs observers design for S-NLPV systems

5.2.1 Preliminaries

Consider the following nonlinear algebro-differential parameter-varying system in its polytopic form.

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[A_i x(t) + B_i u(t) + D_i f(t, F_L x)\Big] + G f_a(t), \\ y(t) &= Cx(t), \end{aligned} \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the known input and $y(t) \in \mathbb{R}^p$ is the measurement output vector and $f_a(t) \in \mathbb{R}^{n_{f_a}}$ is the unknown input vector. Matrix $E \in \mathbb{R}^{n\times n}$ could be singular.

$A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D_i \in \mathbb{R}^{n\times n_f}$ and $G \in \mathbb{R}^{n\times n_{fa}}$, are real matrices and $f(t, F_L x)$ is a nonlinearity satisfying the Lipschitz constraint $||\Delta_f|| \leq \lambda ||F_L(x_1 - x_2)||$ where $\Delta_f = f(t, F_L x_1) - f(t, F_L x_2)$, λ is a known Lipschitz constant and F_L is real matrix of appropriate dimension.
Let $rank(E) = r < n$ and $E^{\perp} \in \mathbb{R}^{s\times n}$ be a full row matrix such that $E^{\perp}E = 0$, in this case $s = n - r$.
Let $\mu_i(\varrho(t))$ be membership functions formed with known variant parameters $\varrho(t) \in \mathbb{R}^l$. The membership functions have the following properties:

$$\sum_{i=1}^{k} \mu_i(\varrho(t)) = 1, \quad \mu_i(\varrho(t)) \geq 0, \tag{5.2}$$

$\forall i = 1\ldots, k$ and $k = 2^l$.
The following definitions and theorem will be used in the sequel of the paper.

Assumption 3. *It is assumed that the system described by equation* (5.1) *is regular (Definition 1), Impulse observable (Definition 2) and Reachable observable (Definition 4).*

Assumption 4. *It is assumed that rank*$(CG) =$ *rank*(G).
This Assumption implies that $p \geq n_{fa}$ the number of measurable outputs must be greater than or equal to the number of faults (Edwards and Tan, 2006).

Assumption 5. *The fault vector is assumed to be constant, i.e.* $\dot{f}_a(t) = 0$.

5.2.2 Problem Statement

Let us consider the following generalized nonlinear observer for system (5.1)

$$\begin{aligned}
\dot{\zeta}(t) &= \sum_{i=1}^{k} \mu_i(\varrho(t)) \Big[N_i(\zeta(t) + TG\hat{f}_a(t)) + H_i v(t) + F_i y(t) + J_i u(t) + \\
&\quad TD_i f(t, F_L \hat{x}) + TG\hat{f}_a(t) \Big], && (5.3)\\
\dot{v}(t) &= \sum_{i=1}^{k} \mu_i(\varrho(t)) \Big[S_i(\zeta(t) + TG\hat{f}_a(t)) + L_i v(t) + M_i y(t) \Big], && (5.4)\\
\hat{x}(t) &= P(\zeta(t) + TG\hat{f}_a(t)) + Qy(t), && (5.5)\\
\dot{\hat{f}}_a(t) &= \Phi_{fa}(C\hat{x}(t) - y(t)), && (5.6)
\end{aligned}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector, $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$ and $\hat{f}_a(t) \in \mathbb{R}^{n_{fa}}$ is the estimate of $f_a(t)$. The matrices $N_i \in \mathbb{R}^{q_0\times q_0}$, $H_i \in \mathbb{R}^{q_0\times q_1}$, $F_i \in \mathbb{R}^{q_0\times p}$, $S_i \in \mathbb{R}^{q_1\times q_0}$, $L_i \in \mathbb{R}^{q_1\times q_1}$, $M_i \in \mathbb{R}^{q_1\times p}$, $J_i \in \mathbb{R}^{q_0\times m}$, $P \in \mathbb{R}^{n\times q_0}$, $Q \in \mathbb{R}^{n\times p}$, and $T \in \mathbb{R}^{q_0\times n}$ are unknown matrices of appropriate dimensions, which must be determined such that $\hat{x}(t)$ converges asymptotically to $x(t)$ and $\hat{f}_a(t)$ converge to $f_a(t)$.
Let a matrix parameter T to define the following transformed error

$$\epsilon(t) = \zeta(t) - TEx(t) + TGf_a(t), \tag{5.7}$$

its derivative is given by

$$\begin{aligned}
\dot{\epsilon}(t) &= \dot{\zeta}(t) - TE\dot{x}(t) \\
\dot{\epsilon}(t) &= \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[N_i((\epsilon(t) + TEx(t) - TGfa(t)) + TG\hat{f}_a(t)) + H_i v(t) + F_i Cx(t) + \\
&\quad Ju(t) + TD_i f(t, F_L\hat{x}) + TG\hat{f}_a(t) - [T(A_i x(t) + B_i u(t) + D_i f(t, F_L x(t)) + Gf_a(t))]\Big] \\
\dot{\epsilon}(t) &= \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[N_i\epsilon(t) + H_i v(t) + (N_i TE - TA_i + F_i C)x(t) + \\
&\quad (J_i - TB_i)u(t) + (N_i TG + TG)e_f(t) + TD_i\Delta_f(t)\Big] \qquad (5.8)
\end{aligned}$$

where $e_f(t) = \hat{f}_a(t) - f_a(t)$ and $\Delta_f(t) = f(t, F_L\hat{x}) - f(t, F_L x)$. Equation (5.8) is linearly independent of $x(t)$ and $u(t)$, if the following equations are satisfied:

$$(a) \quad N_i TE + F_i C - TA_i = 0, \qquad (5.9)$$

$$(b) \quad J_i = TB_i. \qquad (5.10)$$

Then equation (5.8) becomes:

$$\dot{\epsilon}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[N_i\epsilon(t) + H_i v(t) + (N_i TG + TG)e_f(t) + TD_i\Delta_f(t)\Big]. \qquad (5.11)$$

By using equation (5.7), equations (5.4) and (5.5) can be written as

$$\dot{v}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[S_i\epsilon(t) + L_i v(t) + (S_i TE + M_i C)x(t) + S_i TGe_f(t)\Big], \qquad (5.12)$$

$$\hat{x}(t) = P\epsilon(t) + (PTE + QC)x(t) + PTGe_f(t), \qquad (5.13)$$

Now, if the following conditions are satisfied

$$(c) \quad S_i TE + M_i C = 0, \qquad (5.14)$$

$$(d) \quad PTE + QC = I_n, \qquad (5.15)$$

Convergence is ensured if the aforementioned restrictions (a) - (d) are satisfied. However, if these restrictions are not met, the convergence of the observer cannot be guaranteed.

then equation (5.12) becomes

$$\dot{v}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t)\Big[S_i\epsilon(t) + L_i v(t) + S_i TGe_f(t)\Big], \qquad (5.16)$$

and the state estimation error becomes

$$\hat{x}(t) - x(t) = e(t) = P\epsilon(t) + PTGe_f(t). \qquad (5.17)$$

If conditions *(a)–(d)* are satisfied, the following observer error dynamics equation is obtained from (5.11) and (5.16)

$$\dot{\varphi}(t) \;=\; \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[\mathbb{A}_i\varphi(t) + \mathbb{B}_i e_f(t) + \mathbb{C}_i\Delta_f(t)\Big], \tag{5.18}$$

where $\varphi(t) = \begin{bmatrix}\epsilon(t)\\ v(t)\end{bmatrix}$, $\mathbb{A}_i = \begin{bmatrix}N_i & H_i\\ S_i & L_i\end{bmatrix}$, $\mathbb{B}_i = \begin{bmatrix}N_iTG + TG\\ S_iTG\end{bmatrix}$ and $\mathbb{C}_i = \begin{bmatrix}TD_i\\ 0\end{bmatrix}$.
To study the stability of system (5.18) when $e_f(t) = 0$ and $\Delta_f(t) = 0$, let us consider $V(t) = \varphi(t)^T X\varphi(t)$ with $X = X^T > 0$ be a Lyapunov function candidate. Then we have

$$\dot{V}(t) = \dot{\varphi}^T(t)X\varphi(t) + \varphi^T(t)X\dot{\varphi}(t) = \varphi^T(t)[\mathcal{A}^T X + X\mathcal{A}]\varphi(t) \tag{5.19}$$

with $\mathcal{A} = \sum_{i=1}^{k} \mu_i(\varrho(t))\mathbb{A}_i$.
In this case $\dot{V}(t) < 0$ if $\mathcal{A}^T X + X\mathcal{A} < 0$ which equivalent to

$$\sum_{i=1}^{k} \mu_i(\varrho(t))[\mathbb{A}_i^T X + X\mathbb{A}_i] < 0 \tag{5.20}$$

since $\mu_i(\varrho(t)) \geq 0$ and $\sum_{i=1}^{k} = 1$, the stability condition reduces to the one of matrices $\begin{bmatrix}N_i & H_i\\ S_i & L_i\end{bmatrix}$.

5.2.3 Observer design

This section will be devoted to the parameterization of all the matrices of the observer and then the observer design will be formulated as a set of LMIs. Considering a similar parameterization as in Section 4.2.2, it is possible to obtain the values of the unknown matrices of the observer.

Substituting the previous parameterization into the observer error dynamics (5.18), a bilinearity in the product of matrices NTG becomes evident. To address this bilinearity, a modification is made to the parameterization.

Let $\bar{T}_2 = T_2G$ and $Z_1 = Z(I_{n+ny} - \bar{T}_2\bar{T}_2^+)$, where Z is an arbitrary matrix of appropriate dimension, so that, the product of matrices N_iTG becomes

$$N_iTG = N_{1i}T_1G - Z\mathcal{N}_{2i}T_1G - Y_{1i}N_3T_1G,$$

where the fact of $\bar{T}_2\bar{T}_2^+\bar{T}_2 = \bar{T}_2$ is considered. Matrices N_{1i}, N_3, T_1 are previously defined in Section 4.2.2, and matrix $\mathcal{N}_{2i}$ is defined in (5.23).
In the same way, the following expressions are obtained for matrices T, K, N_i and F_i

$$T = T_1 - Z\mathcal{T}_2, \tag{5.21}$$
$$K = K_1 - Z\mathcal{K}_2, \tag{5.22}$$
$$N_i = N_{1i} - Z\mathcal{N}_{2i} - Y_{1i}N_3, \tag{5.23}$$
$$F_i = F_{1i} - Z\mathcal{F}_{2i} - Y_{1i}F_3, \tag{5.24}$$

where $\mathcal{T}_2 = (I_{n+ny} - \bar{T}_2\bar{T}_2^+)T_2$, $\mathcal{K}_2 = (I_{n+ny} - \bar{T}_2\bar{T}_2)K_2$, $\mathcal{N}_{2i} = (I_{n+ny} - \bar{T}_2\bar{T}_2^+)N_{2i}$, $\mathcal{F}_{2i} = (I_{n+ny} - \bar{T}_2\bar{T}_2^+)F_{2i}$ and matrices T_1, T_2, K_1, K_2, N_{1i}, N_{2i}, N_3, F_{1i}, F_{2i} and F_3 are defined in Section 4.2.2. Now, by using the value of matrices N_i, S_i and T given by (5.23), (4.31), (5.21), the observer error dynamics (5.18) can be written as

$$\dot{\varphi}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[(\mathbb{A}_{1i} - \mathbb{Y}_i\mathbb{A}_2)\varphi(t) + (\mathbb{F}_{1i} - \mathbb{Y}_i\mathbb{F}_2)e_f(t) + \mathbb{C}_i\Delta_f(t)\Big], \tag{5.25}$$

where $\mathbb{A}_{1i} = \begin{bmatrix} N_{1i} - Z\mathcal{N}_{2i} & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{F}_{1i} = \begin{bmatrix} N_{1i}T_1G + T_1G - Z\mathcal{N}_{2i}T_1G \\ 0 \end{bmatrix}$, $\mathbb{F}_2 = \begin{bmatrix} N_3T_1G \\ 0 \end{bmatrix}$, $\mathbb{C}_i = \begin{bmatrix} T_1D_i - Z\mathcal{T}_2D_i \\ 0 \end{bmatrix}$ and $\mathbb{Y}_i = \begin{bmatrix} Y_{1i} & H_i \\ Y_{2i} & L_i, \end{bmatrix}$, and from (5.77) we have

$$e(t) = \mathbb{P}\varphi(t) + \mathbb{H}e_f(t), \tag{5.26}$$

where $\mathbb{P} = \begin{bmatrix} P_1 & 0 \end{bmatrix}$ and $\mathbb{H} = P_1T_1G$. Without loss of generality, $Y_{3i} = 0$ is taken for simplicity. Considering Assumption 5 and equation (5.6), $e_f(t)$ can be rewritten as

$$\begin{aligned} \dot{e}_f(t) &= \dot{\hat{f}}_a(t) \\ &= \mathbb{D}\varphi(t) + \mathbb{G}e_f(t) \end{aligned} \tag{5.27}$$

where $\mathbb{D} = \begin{bmatrix} \Phi_{fa}CP_1 & 0 \end{bmatrix}$ and $\mathbb{G} = \Phi_{fa}CP_1T_1G$.
Finally, we can consider $\dot{e}_f(t)$ on the dynamic error equation. Equation (5.25) can be rewritten as

$$\dot{\xi}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[\bar{\mathbb{A}}_i\xi(t) + \bar{\mathbb{C}}_i\Delta_f(t)\Big], \tag{5.28}$$

where $\xi(t) = \begin{bmatrix} \varphi(t) \\ e_f(t) \end{bmatrix}$, $\bar{\mathbb{A}}_i = \begin{bmatrix} \mathbb{A}_{1i} - \mathbb{Y}_i\mathbb{A}_2 & \mathbb{F}_{1i} - \mathbb{Y}_i\mathbb{F}_2 \\ \mathbb{D} & \mathbb{G} \end{bmatrix}$ and $\bar{\mathbb{C}}_i = \begin{bmatrix} \mathbb{C}_i \\ 0 \end{bmatrix}$.
The problem of the observer design is reduced to determine matrices Z and $\mathbb{Y}_i$ such that system (5.28) is asymptotically stable.

5.2.4 Stability analysis of the observer

This section is devoted to the stability analysis of equation (5.28). The following theorem gives the condition for the stability in a set of LMIs.

Theorem 2. *Consider the system given by* (5.3) - (5.6) *as a generalized observer for system* (5.1) *for state estimation if Assumption 3 is fulfilled. The observer* (5.3) - (5.6) *is asymptotically stable if there exists matrices Z, $\mathbb{Y}_i$ and a matrix $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$ with $X_1 = \begin{bmatrix} X_{11} & X_{11} \\ X_{11} & X_{12} \end{bmatrix} > 0$ such that the following LMIs are satisfied.*

$$\mathcal{C}^{T\perp} \begin{bmatrix} \Pi_{1i} + \Pi_{1i}^T + \gamma\lambda^2\mathbb{P}^T F_L^T F_L\mathbb{P} & (*) & (*) \\ \Pi_{2i}^T + X_2\mathbb{D}^T + \gamma\lambda^2\mathbb{H}^T F_L^T F_L\mathbb{P} & \mathbb{G}^T X_2 + X_2\mathbb{G} + \gamma\lambda^2\mathbb{H}^T F_L^T F_L\mathbb{H} & 0 \\ \Pi_{3i}^T & 0 & -\gamma I_{nf} \end{bmatrix} \mathcal{C}^{T\perp T} < 0 \tag{5.29}$$

$$\begin{bmatrix} \mathbb{G}^T X_2 + X_2\mathbb{G} + \gamma\lambda^2\mathbb{H}^T F_L^T F_L \mathbb{H} & 0 \\ 0 & -\gamma I_{nf} \end{bmatrix} < 0, \tag{5.30}$$

where the symbol $(*)$ *denotes the transpose elements on the symmetric positions,*

$$\mathcal{C}^{T\perp} = \begin{bmatrix} \mathbb{A}_2^{T\perp} & 0 & 0 \\ 0 & \mathbb{F}_2^{T\perp} & 0 \\ 0 & 0 & I \end{bmatrix}, \Pi_{1i} = \begin{bmatrix} X_{11}N_{1i} - X_z\mathcal{N}_{2i} & 0 \\ X_{11}N_{1i} - X_z\mathcal{N}_{2i} & 0 \end{bmatrix}, \tag{5.31}$$

$$\Pi_{2i} = \begin{bmatrix} X_{11}N_{1i}T_1G + X_{11}T_1G - X_z\mathcal{N}_{2i}T_1G \\ X_{11}N_{1i}T_1G + X_{11}T_1G - X_z\mathcal{N}_{2i}T_1G \end{bmatrix}, \Pi_3 = \begin{bmatrix} X_{11}T_1D - X_z\mathcal{T}_2D \\ X_{11}T_1D - X_z\mathcal{T}_2D \end{bmatrix}, \tag{5.32}$$

in this case matrix $Z = X_{11}^{-1}X_z$ *and parameter matrix* Φ_i^T *is obtained as follows*

$$\Phi_i^T = \mathcal{B}_r{}^+\mathcal{K}_i\mathcal{C}_l{}^+ + \mathcal{Z} - \mathcal{B}_r{}^+\mathcal{B}_r\mathcal{Z}\mathcal{C}_l\mathcal{C}_l{}^+, \tag{5.33}$$

where

$$\mathbb{Y}_i = (\Phi_i X_1^{-1})^T, \tag{5.34}$$

$$\mathcal{K}_i = -\mathcal{R}^{-1}\mathcal{B}_l{}^T\mathcal{V}_i\mathcal{C}_r{}^T(\mathcal{C}_r\mathcal{V}_i\mathcal{C}_r{}^T)^{-1} + \mathcal{S}_i^{\frac{1}{2}}\mathcal{L}(\mathcal{C}_r\mathcal{V}_i\mathcal{C}_r{}^T)^{-\frac{1}{2}}, \tag{5.35}$$

$$\mathcal{S}_i = \mathcal{R}^{-1} - \mathcal{R}^{-1}\mathcal{B}_l^T[\mathcal{V}_i - \mathcal{V}_i\mathcal{C}_r^T(\mathcal{C}_r\mathcal{V}_i\mathcal{C}_r{}^T)^{-1}\mathcal{C}_r\mathcal{V}_i]\mathcal{B}_l\mathcal{R}^{-1}, \tag{5.36}$$

$$\mathcal{V}_i = (\mathcal{B}_r\mathcal{R}^{-1}\mathcal{B}_l{}^T - \mathcal{D}_i)^{-1} > 0, \tag{5.37}$$

where matrices $\mathcal{Z}$, $\mathcal{L}$, $\mathcal{R}$ *are arbitrary matrices such that* $||\mathcal{L}|| < 1$ *and* $\mathcal{R} > 0$, *with*

$$\mathcal{D}_i = \begin{bmatrix} \Pi_{1i} + \Pi_{1i}^T + \gamma\lambda^2\mathbb{P}^T F_L^T F_L\mathbb{P} & (*) & (*) \\ \Pi_{2i}^T + X_2\mathbb{D}^T + \gamma\lambda^2\mathbb{H}^T F_L^T F_L\mathbb{P} & \mathbb{G}^T X_2 + X_2\mathbb{G} + \gamma\lambda^2\mathbb{H}^T F_L^T F_L\mathbb{H} & 0 \\ \Pi_{3i}^T & 0 & -\gamma I_{nf} \end{bmatrix},$$

$\mathcal{B} = \begin{bmatrix} -I \\ 0 \\ 0 \end{bmatrix}$ *and* $\mathcal{C} = \begin{bmatrix} \mathbb{A}_2 & \mathbb{F}_2 & 0 \end{bmatrix}$, *such that there exist matrices* $\mathcal{B}_l$, $\mathcal{B}_r$, $\mathcal{C}_l$ *and* $\mathcal{C}_r$, *are such that* $\mathcal{B} = \mathcal{B}_l\mathcal{B}_r$ *and* $\mathcal{C} = \mathcal{C}_l\mathcal{C}_r$, *respectively.*

Proof. By considering the following Lyapunov function (Wu et al., 2013)

$$V(\xi(t)) = \xi(t)^T X\xi(t), \tag{5.38}$$

such that

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, \tag{5.39}$$

with $X_1 = X_1^T = \begin{bmatrix} X_{11} & X_{11} \\ X_{11} & X_{12} \end{bmatrix} > 0$.

The derivative of (5.38) along the trajectory of (5.28) gives

$$\dot{V}(\xi(t)) = \sum_{i=1}^{k}\mu_i(\varrho(t))\Big[\xi^T(t)(\bar{\mathbb{A}}_i^T X + X^T\bar{\mathbb{A}}_i)\xi(t) + \xi^T(t)X^T\bar{\mathbb{C}}_i\Delta_f(t) + \Delta_f^T(t)\bar{\mathbb{C}}_i^T X\xi(t)\Big], \tag{5.40}$$

By using Lemma 3 we can obtain the following inequality

$$\Delta_f^T(t)\bar{\mathbb{C}}_i^T X\xi(t) + \xi^T(t)X^T\bar{\mathbb{C}}_i\Delta_f(t) \le \gamma\Delta_f^T(t)\Delta_f(t) + \frac{1}{\gamma}\xi^T(t)X^T\bar{\mathbb{C}}_i\bar{\mathbb{C}}_i^T X\xi(t), \tag{5.41}$$

and from the Lipschitz condition, we have

$$\Delta_f^T\Delta_f \le \lambda^2\xi^T(t)\begin{bmatrix}\mathbb{P}^T F_L^T F_L\mathbb{P} & \mathbb{P}^T F_L^T F_L\mathbb{H}\\ \mathbb{H}^T F_L^T F_L\mathbb{P} & \mathbb{H}^T F_L^T F_L\mathbb{H},\end{bmatrix}\xi(t), \tag{5.42}$$

by inserting (5.41) and (5.42) into (5.40), we obtain

$$\dot{V}(\xi(t)) \le \sum_{i=1}^{k}\mu_i(\varrho(t))\left[\xi^T(t)\left(\bar{\mathbb{A}}_i^T X + X^T\bar{\mathbb{A}}_i + \frac{1}{\gamma}X^T\bar{\mathbb{C}}_i\bar{\mathbb{C}}_i^T X + \gamma\lambda^2\begin{bmatrix}\mathbb{P}^T F_L^T F_L\mathbb{P} & \mathbb{P}^T F_L^T F_L\mathbb{H}\\ \mathbb{H}^T F_L^T F_L\mathbb{P} & \mathbb{H}^T F_L^T F_L\mathbb{H},\end{bmatrix}\right)\xi(t)\right].$$

Now, if the following LMI is satisfied then $\dot{V}(\xi(t)) < 0$

$$\bar{\mathbb{A}}_i^T X + X^T\bar{\mathbb{A}}_i + \frac{1}{\gamma}X^T\bar{\mathbb{C}}_i\bar{\mathbb{C}}_i^T X + \gamma\lambda^2\begin{bmatrix}\mathbb{P}^T F_L^T F_L\mathbb{P} & \mathbb{P}^T F_L^T F_L\mathbb{H}\\ \mathbb{H}^T F_L^T F_L\mathbb{P} & \mathbb{H}^T F_L^T F_L\mathbb{H},\end{bmatrix} < 0, \tag{5.43}$$

by using the Schur complement we obtain

$$\begin{bmatrix}\bar{\mathbb{A}}_i^T X + X^T\bar{\mathbb{A}}_i + \gamma\lambda^2\begin{bmatrix}\mathbb{P}^T F_L^T F_L\mathbb{P} & \mathbb{P}^T F_L^T F_L\mathbb{H}\\ \mathbb{H}^T F_L^T F_L\mathbb{P} & \mathbb{H}^T F_L^T F_L\mathbb{H},\end{bmatrix} & (*)\\ \bar{\mathbb{C}}_i^T X & -\gamma I_{nf}\end{bmatrix} < 0. \tag{5.44}$$

By replacing matrices $\bar{\mathbb{A}}_i$, $\bar{\mathbb{C}}_i$ from (5.25), and X from (5.39) in inequality (5.44), we obtain

$$\begin{bmatrix}\Pi_{1i} + \Pi_{1i}^T - \mathbb{A}_2^T\Phi_i - \Phi_i^T\mathbb{A}_2 + \gamma\lambda^2\mathbb{P}^T F_L^T F_L\mathbb{P} & (*) & (*)\\ \Pi_{2i}^T + X_2\mathbb{D}^T - \mathbb{F}_2^T\Phi_i + \gamma\lambda^2\mathbb{H}^T F_L^T F_L\mathbb{P} & \mathbb{G}^T X_2 + X_2\mathbb{G} + \gamma\lambda^2\mathbb{H}^T F_L^T F_L\mathbb{H} & 0\\ \mathbb{C}_i^T X_1 & 0 & -\gamma I_{nf}\end{bmatrix} < 0, \tag{5.45}$$

where $\Phi_i = \mathbb{Y}_i^T X_1$.
Now, by inserting the values of matrices $\mathbb{A}_{1i}$ and X_{11} we obtain

$$\begin{bmatrix}\Pi_{1i} + \Pi_{1i}^T - \mathbb{A}_2^T\Phi_i - \Phi_i^T\mathbb{A}_2 + \gamma\lambda^2\mathbb{P}^T F_L^T F_L\mathbb{P} & (*) & (*)\\ \Pi_{2i}^T + X_2\mathbb{D}^T - \mathbb{F}_2^T\Phi_i + \gamma\lambda^2\mathbb{H}^T F_L^T F_L\mathbb{P} & \mathbb{G}^T X_2 + X_2\mathbb{G} + \gamma\lambda^2\mathbb{H}^T F_L^T F_L\mathbb{H} & 0\\ \Pi_{3i}^T & 0 & -\gamma I_{nf}\end{bmatrix} < 0, \tag{5.46}$$

where Π_{1i}, Π_{2i} and Π_{3i} are defined in (5.32), and $X_z = X_{11}Z$.
Inequality (5.46) can also be rewritten as

$$(\mathcal{B}\Phi_i^T\mathcal{C}) + (\mathcal{B}\Phi_i^T\mathcal{C})^T + \mathcal{D}_i < 0, \tag{5.47}$$

where

$$\mathcal{D}_i = \begin{bmatrix}\Pi_{1i} + \Pi_{1i}^T + \gamma\lambda^2\mathbb{P}^T F_L^T F_L\mathbb{P} & (*) & (*)\\ \Pi_{2i}^T + X_2\mathbb{D}^T + \gamma\lambda^2\mathbb{H}^T F_L^T F_L\mathbb{P} & \mathbb{G}^T X_2 + X_2\mathbb{G} + \gamma\lambda^2\mathbb{H}^T F_L^T F_L\mathbb{H} & 0\\ \Pi_{3i}^T & 0 & -\gamma I_{nf}\end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} -I \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathcal{C} = \begin{bmatrix} \mathbb{A}_2 & \mathbb{F}_2 & 0 \end{bmatrix}.$$

According to Lemma 2, inequality (5.47) is satisfied if and only if the following inequalities are verified

$$\mathcal{C}^{T\perp}\mathcal{D}_i\mathcal{C}^{T\perp T} < 0, \tag{5.48}$$

$$\mathcal{B}^{\perp}\mathcal{D}_i\mathcal{B}^{\perp T} < 0, \tag{5.49}$$

the inequalities (5.48) and (5.49) are equivalent to (5.29) and (5.30) respectively

$$\mathcal{C}^{T\perp}\begin{bmatrix} \Pi_{1i} + \Pi_{1i}^T + \gamma\lambda^2\mathbb{P}^T F_L^T F_L \mathbb{P} & (*) & (*) \\ \Pi_{2i}^T + X_2\mathbb{D}^T + \gamma\lambda^2\mathbb{H}^T F_L^T F_L\mathbb{P} & \mathbb{G}^T X_2 + X_2\mathbb{G} + \gamma\lambda^2\mathbb{H}^T F_L^T F_L\mathbb{H} & 0 \\ \Pi_{3i}^T & 0 & -\gamma I_{nf} \end{bmatrix}\mathcal{C}^{T\perp T} < 0, \tag{5.50}$$

$$\begin{bmatrix} \mathbb{G}^T X_2 + X_2\mathbb{G} + \gamma\lambda^2\mathbb{H}^T F_L^T F_L\mathbb{H} & 0 \\ 0 & -\gamma I_{nf} \end{bmatrix} < 0. \tag{5.51}$$

Then by using the results of Lemma 2 we obtain the solution of $\mathbb{Y}_i$ by using (5.33)-(5.37) which complete the proof. □

The following algorithm summarizes the generalized nonlinear observer design procedure.

1. Select the observer order q_0 and define matrix $R \in \mathbb{R}^{q_0 \times n}$ such that $\text{rank}(\Sigma) = n$.

2. Compute matrices N_{1i}, $\mathcal{N}_{2i}$, N_3, T_1, $\mathcal{T}_2$, K_1, $\mathcal{K}_2$ and P_1 defined in Section 5.2.3.

3. Solve LMIs (5.29) and (5.30) to find X, Z and $\gamma > 0$.

4. Find matrices $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{Z}$, such that $||\mathcal{L}|| < 1$, $\mathcal{R} > 0$ and $\mathcal{V}_i > 0$.

5. Considering the elimination lemma, determine the parameter matrix determine the parameter matrix $\mathbb{Y}_i$ as in equation (5.34) which involves the unknown observer matrices.

6. Compute all the matrices of the generalized nonlinear observer (5.3)-(5.5); N_i, H_i, F_i, J_i, S_i, L_i, M_i, P and Q, by using (5.23) to compute N_i, (5.34) to compute H_i and L_i, (4.31)-(4.34) to compute S_i, M_i, P and Q taking matrix $Y_{3i} = 0$. The matrices F_i are given by (5.24) and matrix J_i from (5.10).

5.2.5 Application to heat exchanger

A heat exchanger with two countercurrent cells is a specific type of heat exchanger that features two countercurrent cells. In this device, two streams of fluid (typically, a hot fluid and a cold fluid) flow in opposite directions within the heat exchanger.

The countercurrent configuration is common in heat exchangers because it maximizes heat transfer efficiency. In this design, the temperature difference between the two fluids is maintained high across the entire heat exchanger, resulting in enhanced heat transfer. Additionally, at the end of each countercurrent cell, the temperature of the hot fluid is lowest and that of the cold fluid is highest, further maximizing thermal efficiency. The presence of two countercurrent cells in the heat exchanger provides a larger heat transfer area and thus greater heat transfer capacity compared to single-cell heat exchangers. This enables more efficient heat exchange and greater flexibility in the application of the heat exchanger in various industrial processes, air conditioning systems, refrigeration, and other applications where heat transfer is required.

An observer is needed to estimate faults and states in a heat exchanger due to the difficulty of directly measuring certain critical variables, such as temperature and fluid flow, as well as the presence of potential faults or deterioration in the system. The observer can use mathematical models and input data to accurately estimate the current state of the heat exchanger and detect possible faults or anomalies before they significantly affect its performance. This allows for continuous monitoring and preventive action to maintain the efficiency and reliability of the heat exchanger (Thulukkanam, 2000).

Model of the heat exchanger with two countercurrent cells

To show the effectiveness of the observer, we apply our results to a heat exchanger with two countercurrent cells (Figure 5.1) (Dobos et al., 2009).

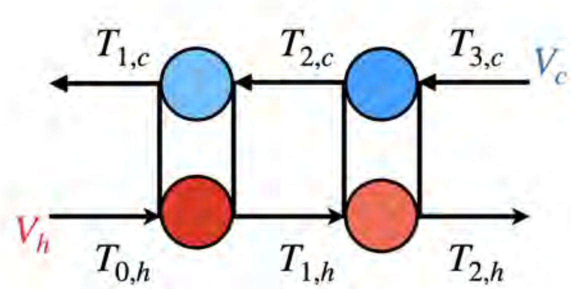

Fig. 5.1. Schematic representation of the double-pipe heat exchanger.

The equations that represent the energy balance are given in (5.52)

$$
\begin{aligned}
\frac{dT_{1,c}}{dt} =& \frac{2v_c}{V_c}(T_{2,c} - T_{1,c}) + \frac{UA}{\rho_c C\rho_c V_c}\Delta T_1 \\
\frac{dT_{1,h}}{dt} =& \frac{2v_h}{V_h}(T_{0,h} - T_{1,h}) - \frac{UA}{\rho_h C\rho_h V_h}\Delta T_1 \\
\frac{dT_{2,c}}{dt} =& \frac{2v_c}{V_c}(T_{3,c} - T_{2,c}) + \frac{UA}{\rho_c C\rho_c V_c}\Delta T_2 \\
\frac{dT_{2,h}}{dt} =& \frac{2v_h}{V_h}(T_{1,h} - T_{2,h}) - \frac{UA}{\rho_h C\rho_h V_h}\Delta T_2
\end{aligned}
\tag{5.52}
$$

where ΔT_1 and ΔT_2 are defined as follows

$$
\begin{aligned}
\Delta T_1 =& \frac{(T_{1,h} - T_{2,c}) - (T_{0,h} - T_{1,c})}{ln\left(\dfrac{(T_{1,h} - T_{2,c})}{(T_{0,h} - T_{1,c})}\right)} \\
\Delta T_2 =& \frac{[T_{2,h} - T_{3,c}] - [T_{1,h} - T_{2,c}]}{ln\left[\dfrac{[T_{2,h} - T_{3,c}]}{[T_{1,h} - T_{2,c}]}\right]}
\end{aligned}
\tag{5.53}
$$

The rate of heat flow across the solid-fluid interface is $Q = UA\Delta T_1$ and $Q_2 = UA\Delta T_2$.

V_c is the volume in external side ($134.99 \times 10^{-6} m^3$), V_h is the Volume in the inner side ($15.512 \times 10^{-6} m^3$), v_c is the Flow tn the cold stream ($6.399 \times 10^{-6}\text{cm}^3/\text{min}$), v_h is the Flow in the hot stream ($1.94 \times 10^{-5}\text{cm}^3/\text{min}$), $C\rho_c$ is the Specific heat of cold water (4181.5 J/Kg°C), $C\rho_h$ is the Specific heat of hot water (4196.5 J/Kg°C), ρ_c is the Density of cold water (996.781Kg/m^3), ρ_h is the Density of hot water (971.150Kg/m^3), A is the Heat transfer surface area (0.015387511m^2) and U is the Global heat transfer coefficient (1400W/m^2).

$T_{3,c}(t)$ and $T_{0,h}(t)$ are the inlet temperatures in the cold and hot streams respectively. $T_{1,c}(t)$ and $T_{2,h}(t)$ are the outlet temperatures in the cold and hot streams, respectively.

Nonlinear algebro-differential parameter-varying system formulation

Considering the model presented in Equation (5.52), the following nonlinear state space representation is obtained:

$$
\underbrace{\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}}_{E}\underbrace{\begin{bmatrix}\dot{T}_{1,c}(t)\\\dot{T}_{1,h}(t)\\\dot{T}_{2,c}(t)\\\dot{T}_{2,h}(t)\\\dot{Q}(t)\\\dot{Q}_2(t)\end{bmatrix}}_{\dot{x}(t)}=\underbrace{\begin{bmatrix}-\frac{2v_c}{V_c}&0&\frac{2v_c}{V_c}&0&\frac{1}{C\rho_c\rho_cV_c}&0\\0&-\frac{2v_h(t)}{V_h}&0&0&-\frac{1}{C\rho_c\rho_cV_c}&0\\0&0&-\frac{2v_c}{V_c}&0&0&\frac{1}{C\rho_c\rho_cV_c}\\0&\frac{2v_h(t)}{V_h}&0&-\frac{2v_h(t)}{V_h}&0&-\frac{1}{C\rho_c\rho_cV_c}\\0&0&0&0&-\frac{1}{UA}&0\\0&0&0&0&0&-\frac{1}{UA}\end{bmatrix}}_{A(\rho(t))}\underbrace{\begin{bmatrix}T_{1,c}(t)\\T_{1,h}(t)\\T_{2,c}(t)\\T_{2,h}(t)\\Q(t)\\Q_2(t)\end{bmatrix}}_{x(t)}+
$$

$$
\underbrace{\begin{bmatrix}0&0\\\frac{2v_h}{V_h}&0\\0&\frac{2v_c}{V_c}\\0&0\\0&0\\0&0\end{bmatrix}}_{B}\underbrace{\begin{bmatrix}T_{0,h}\\T_{3,c}\end{bmatrix}}_{u(t)}+\underbrace{\begin{bmatrix}0\\0\\\frac{2v_c}{V_c}\\0\\0\\0\end{bmatrix}}_{G}f_a(t)+\underbrace{\begin{bmatrix}0&0\\0&0\\0&0\\0&0\\1&0\\0&1\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}\Delta T_1\\\Delta T_2\end{bmatrix}}_{f(t,F_Lx)}, \tag{5.54}
$$

$$
y(t)=\underbrace{\begin{bmatrix}0&0&1&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}}_{C}x(t), \tag{5.55}
$$

where the flow the cold stream $\rho(t) = v_h(t)$ is considered as the time varying parameter, and the nonlinearity $f(t, F_Lx)$ is considered as a Lipschitz function. The input signal is considered constant $u(t) = \begin{bmatrix}T_{0,h} & T_{3,c}\end{bmatrix} = \begin{bmatrix}77.9°C & 21.2°C\end{bmatrix}$.
The fault signal is defined as

$$
f(t) = \begin{cases} 8°C & 80s \leq t \leq 120s \\ 0°C & \text{otherwise.} \end{cases} \tag{5.56}
$$

with this, the Lipschitz constant is selected as $\lambda = 100$. In fact the states $Q(t)$ and $Q_2(t)$ are in a bounded set and ΔT_1, ΔT_2 are Lipschitz functions, as shown below.

Determination of the Lipschitz constant

First we recall that if a function $f(x, u, t)$ is continuous in a closed and bounded region D and if $\dfrac{\partial f(x,u,t)}{\partial x}$ is continuous in D, then there exist a positive constant λ such that $\|f(x_1, u, t) - f(x_2, u, t)\| \leq \lambda \|x_1 - x_2\|$. Now, to calculate this constant for our model, we can put the Jacobian matrix of the system (5.54)-(5.55) as shown below

$$
\frac{\partial f(x,u,t)}{\partial x(t)} = \begin{bmatrix}
-\frac{2v_c}{V_c} + UA\frac{\partial \Delta T_1}{\partial T_{1,c}} & UA\frac{\partial \Delta T_1}{\partial T_{1,h}} & \frac{2v_c}{V_c} + UA\frac{\partial \Delta T_1}{\partial T_{2,c}} & 0 & \frac{1}{C\rho_c\rho_c V_c} & 0 \\
UA\frac{\partial \Delta T_1}{\partial T_{1,c}} & -\frac{2v_h(t)}{V_h} + UA\frac{\partial \Delta T_1}{\partial T_{1,h}} & UA\frac{\partial \Delta T_1}{\partial T_{2,c}} & 0 & -\frac{1}{C\rho_h\rho_h V_h} & 0 \\
0 & UA\frac{\partial \Delta T_2}{\partial T_{1,h}} & -\frac{2v_c}{V_c} + UA\frac{\partial \Delta T_2}{\partial T_{2,c}} & UA\frac{\partial \Delta T_2}{\partial T_{2,h}} & 0 & \frac{1}{C\rho_c\rho_c V_c} \\
0 & \frac{2v_h(t)}{V_h} + UA\frac{\partial \Delta T_2}{\partial T_{1,h}} & UA\frac{\partial \Delta T_2}{\partial T_{2,c}} & -\frac{2v_h(t)}{V_h} + UA\frac{\partial \Delta T_2}{\partial T_{2,h}} & 0 & -\frac{1}{C\rho_h\rho_h V_h} \\
0 & 0 & 0 & 0 & -\frac{1}{UA} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{UA}
\end{bmatrix} \tag{5.57}
$$

since in a finite space all the norms are equivalent, the norm 1 is used in (5.57) to obtain the maximum column value, resulting in

$$
\begin{aligned}
\left\|\frac{\partial f(x,u,t)}{\partial x(t)}\right\|_1 \leq max \Bigg\{ & \left|-\frac{2v_c}{V_c} + UA\frac{\partial \Delta T_1}{\partial T_{1,c}}\right| + \left|UA\frac{\partial \Delta T_1}{\partial T_{1,c}}\right|, \left|UA\frac{\partial \Delta T_1}{\partial T_{1,h}}\right| + \\
& \left|-\frac{2v_h(t)}{V_h} + UA\frac{\partial \Delta T_1}{\partial T_{1,h}}\right| + \left|UA\frac{\partial \Delta T_2}{\partial T_{1,h}}\right| + \left|\frac{2v_h(t)}{V_h} + UA\frac{\partial \Delta T_2}{\partial T_{1,h}}\right|, \\
& \left|\frac{2v_c}{V_c} + UA\frac{\partial \Delta T_1}{\partial T_{2,c}}\right| + \left|UA\frac{\partial \Delta T_1}{\partial T_{2,c}}\right| + \left|-\frac{2v_c}{V_c} + UA\frac{\partial \Delta T_2}{\partial T_{2,c}}\right| + \\
& \left|UA\frac{\partial \Delta T_2}{\partial T_{2,c}}\right|, \left|UA\frac{\partial \Delta T_2}{\partial T_{2,h}}\right| + \left|-\frac{2v_h(t)}{V_h} + UA\frac{\partial \Delta T_2}{\partial T_{2,h}}\right|, \\
& \left|\frac{1}{C\rho_c\rho_c V_c}\right| + \left|-\frac{1}{C\rho_h\rho_h V_h}\right| + \left|-\frac{1}{UA}\right|, \\
& \left|\frac{1}{C\rho_c\rho_c V_c}\right| + \left|-\frac{1}{C\rho_h\rho_h V_h}\right| + \left|-\frac{1}{UA}\right| \Bigg\}
\end{aligned} \tag{5.58}
$$

substituting the parameters values of the heat model of Section 5.2.5 in the equation (5.58) we obtain

$$
\left\|\frac{\partial f(x,u,t)}{\partial x(t)}\right\|_1 \leq max\left\{22.81, 46.13, 41.33, 14.76, 0.064, 0.064\right\} \tag{5.59}
$$

Therefore any constant greater than the maximum value of the equation (5.59) can be selected as the Lipschitz constant λ for this particular example.
The maximum value of (5.59) is then obtained as :

$$
\left\|\frac{\partial f(x,u,t)}{\partial x(t)}\right\|_1 \leq max\left\{22.81, 46.13, 41.33, 14.76, 0.064, 0.064\right\} = 46.13 \leq \lambda \tag{5.60}
$$

which justify the choice of $\lambda = 100$.

5.2.6 Simulation results

First we can see that Assumptions 3, 4 and 5 are satisfied, then we can apply the algorithm of Section 5.2.4 as follows

1. By fixing $q_0 = 6$, we can choose matrix $R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times 10^3$, such that

$rank(\Sigma) = rank \begin{bmatrix} R \\ C \end{bmatrix} = 6.$

2. We obtain the following matrices $N_{11} = \begin{bmatrix} -2.36 & -2.37 & 0 & 0 & 2.36 & 0 \\ -2.36 & -2.37 & 0 & 0 & 2.36 & 0 \\ 2.37 & 2.39 & -2.37 & 0 & -2.39 & 0 \\ 0 & 0 & 0 & -0.05 & 0 & 0 \\ -2.36 & -2.37 & 0 & 0 & 2.36 & 0 \\ 0 & -0.09 & 0 & 0.09 & 0.10 & -0.09 \end{bmatrix}$,

$N_{12} = \begin{bmatrix} -2.49 & -2.50 & 0 & 0 & 2.49 & 0 \\ -2.49 & -2.50 & 0 & 0 & 2.49 & 0 \\ 2.50 & 2.52 & -2.50 & 0 & -2.52 & 0 \\ 0 & 0 & 0 & -0.05 & 0 & 0 \\ -2.49 & -2.50 & 0 & 0 & 2.49 & 0 \\ 0 & -0.09 & 0 & 0.09 & 0.10 & -0.09 \end{bmatrix}$, $T_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times 10^3$,

$T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.50 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.50 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $K_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \times 10^3$, $K_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \end{bmatrix} \times 10^{-3}$, $N_{21} = N_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.089 & 0 & -0.47 & 0.0089 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.4642 & 0 & 0 & 0 & 0.4642 & 0 \\ 0 & -0.4642 & 0 & 0 & 0.4642 & 0 \\ 0 & 0.0089 & 0 & 0.4740 & -0.0089 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times 10^{-4}$,

$$N_3 = \begin{bmatrix} 0.0010 & 0 & 0 & 0 & -0.0010 & 0 \\ 0 & 0.0010 & 0 & 0 & -0.0010 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0010 & 0 & 0 \\ -0.0010 & -0.0010 & 0 & 0 & 0.0020 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix} \times 10^{-3}.$$

3. By using the YALMIP toolbox Lofberg (2004) we solve LMIs (5.29) and (5.30) to obtain

$$X = \begin{bmatrix} 4.07 & -1.36 & 0.39 & 0 & -1.36 & 0 & 4.07 & -1.36 & 0.39 & 0 & -1.36 & 0 & 0 \\ -1.36 & 4.07 & 0.39 & 0 & -1.36 & 0 & -1.36 & 4.07 & 0.39 & 0 & -1.36 & 0 & 0 \\ 0.39 & 0.39 & 2.59 & 0 & 0.39 & 0 & 0.39 & 0.39 & 2.59 & 0 & 0.39 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.36 & -1.36 & 0.39 & 0 & 4.07 & 0 & -1.36 & -1.36 & 0.39 & 0 & 4.07 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 14.92 & 0 & 0 & 0 & 0 & 0 & 14.92 & 0 \\ 4.07 & -1.36 & 0.39 & 0 & -1.36 & 0 & 15.05 & -1.24 & 0.14 & 0 & -1.24 & 0 & 0 \\ -1.36 & 4.07 & 0.39 & 0 & -1.36 & 0 & -1.24 & 15.05 & 0.14 & 0 & -1.24 & 0 & 0 \\ 0.39 & 0.39 & 2.59 & 0 & 0.39 & 0 & 0.14 & 0.14 & 12.96 & 0 & 0.14 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 11.87 & -0 & 0 & 0 \\ -1.36 & -1.36 & 0.39 & 0 & 4.07 & 0 & -1.24 & -1.24 & 0.14 & 0 & 15.05 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 14.92 & 0 & 0 & 0 & 0 & 0 & 26.70 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.10 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0.73 & 0.73 & 0 & 0.73 & 0 & 0 & 0 & 0 & 0.73 \\ 0.73 & 0.73 & 0 & 0.73 & 0 & 0 & 0 & 0 & 0.73 \\ 0.05 & 0.05 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0.05 \\ 361.34 & 361.34 & 0 & 361.34 & 0 & 0 & 0 & 0 & 361.34 \\ 0.73 & 0.73 & 0 & 0.73 & 0 & 0 & 0 & 0 & 0.73 \\ 0.07 & 0.07 & 0 & 0.07 & 0 & 0 & 0 & 0 & 0.07 \end{bmatrix}, \text{ and } \gamma = 11.87.$$

4. Considering $\mathcal{R} = I_{12} \times 10$ and $\mathcal{L} = ones_{12,9} \times 0.1$, satisfying the following conditions $||\mathcal{L}|| < 1$, and $\mathcal{R} > 0$ such that $\mathcal{V}_i > 0$.
5. Taking matrix $\mathcal{Z} = 0$ we obtain matrix $\mathbb{Y}_i$ as in equation (5.34).
6. We deduce all the matrices of the generalized nonlinear observer and we obtain

$$N_1 = \begin{bmatrix} -19.54 & 8.08 & 0 & 0.01 & 9.08 & 0 \\ 8.09 & -19.54 & 0 & 0.01 & 9.08 & 0 \\ 0.79 & 0.79 & -2.37 & 0 & 0.79 & 0 \\ -15.36 & -15.54 & 0 & -1.61 & 30.90 & 0 \\ 8.09 & 8.08 & 0 & 0.01 & -18.54 & 0 \\ 0.35 & 0.47 & 0 & -0.12 & -0.82 & -0.09 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} -19.61 & 8.01 & 0 & 0.01 & 9.10 & 0 \\ 8.02 & -19.61 & 0 & 0.01 & 9.10 & 0 \\ 0.83 & 0.83 & -2.50 & 0 & 0.85 & 0 \\ -15.37 & -15.56 & 0 & -1.61 & 30.93 & 0 \\ 8.02 & 8.01 & 0 & 0.01 & -18.53 & 0 \\ 0.35 & 0.47 & 0 & -0.12 & -0.83 & -0.09 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 9.06 & -0.15 & -2.08 & -0.61 & -0.15 & -0.61 \\ -0.15 & 9.06 & -2.08 & -0.61 & -0.15 & -0.61 \\ 0.47 & 0.47 & 13.10 & 0.07 & 0.47 & 0.07 \\ -26.52 & -26.52 & -25.30 & -16.62 & -26.52 & -25.91 \\ -0.15 & -0.15 & -2.08 & -0.61 & 9.06 & -0.61 \\ 0.61 & 0.61 & 0.58 & 0.60 & 0.61 & 9.29 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 9.05 & -0.16 & -2.20 & -0.65 & -0.16 & -0.65 \\ -0.16 & 9.05 & -2.20 & -0.65 & -0.16 & -0.65 \\ 0.46 & 0.46 & 13.24 & 0.05 & 0.46 & 0.05 \\ -26.54 & -26.54 & -25.26 & -16.62 & -26.54 & -25.91 \\ -0.16 & -0.16 & -2.20 & -0.65 & 9.05 & -0.65 \\ 0.61 & 0.61 & 0.58 & 0.60 & 0.61 & 9.29 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 5.59 & -3.62 & 0 & 0.01 & -1.97 & 0 \\ -3.62 & 5.58 & 0 & 0.01 & -1.97 & 0 \\ -0.23 & -0.24 & 0 & 0.01 & 0.47 & 0 \\ -0.49 & -0.50 & 0 & 0.01 & 0.99 & 0 \\ -3.62 & -3.62 & 0 & 0.01 & 7.24 & 0 \\ -0.42 & -0.54 & 0 & 0.13 & 0.96 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} -4.06 & -15.82 & 0 \\ -4.06 & -15.82 & 0 \\ 0.48 & 0 & -15.82 \\ 757.58 & 0 & 0.89 \\ -4.06 & 0 & 0 \\ 155.84 & 0 & 0 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 5.59 & -3.63 & 0 & 0.01 & -1.96 & 0 \\ -3.62 & 5.58 & 0 & 0.01 & -1.96 & 0 \\ -0.22 & -0.22 & 0 & 0.01 & 0.45 & 0 \\ -0.49 & -0.50 & 0 & 0.01 & 0.99 & 0 \\ -3.62 & -3.63 & 0 & 0.01 & 7.25 & 0 \\ -0.42 & -0.54 & 0 & 0.13 & 0.96 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -4.33 & -15.82 & 0 \\ -4.33 & -15.82 & 0 \\ 0.34 & 0 & -15.82 \\ 757.58 & 0 & 0.89 \\ -4.33 & -15.82 & 0 \\ 155.84 & 1.78 & 0 \end{bmatrix},$$

$$J = \begin{bmatrix} 2372.36 & 0 \\ 2372.36 & 0 \\ 0 & 0 \\ 0 & 47.40 \\ 2372.36 & 0 \\ 0 & 0 \end{bmatrix}, \quad L_1 = \begin{bmatrix} -9.91 & -0.71 & -0.84 & -0.72 & -0.71 & -0.72 \\ -0.71 & -9.91 & -0.84 & -0.72 & -0.71 & -0.72 \\ -1.03 & -1.03 & -10.87 & -0.71 & -1.03 & -0.70 \\ -0.85 & -0.85 & -0.81 & -9.27 & -0.85 & -0.83 \\ -0.71 & -0.71 & -0.84 & -0.72 & -9.91 & -0.72 \\ -0.72 & -0.72 & -0.69 & -0.71 & -0.72 & -9.31 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} -9.91 & -0.71 & -0.83 & -0.72 & -0.71 & -0.72 \\ -0.71 & -9.91 & -0.83 & -0.72 & -0.71 & -0.72 \\ -1.02 & -1.02 & -10.88 & -0.70 & -1.02 & -0.70 \\ -0.85 & -0.85 & -0.81 & -9.27 & -0.85 & -0.83 \\ -0.71 & -0.71 & -0.83 & -0.72 & -9.91 & -0.72 \\ -0.72 & -0.72 & -0.69 & -0.71 & -0.72 & -9.31 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \end{bmatrix} \times 10^{-3},$$

$$M_1 = \begin{bmatrix} -4.79 & 0 & 0 \\ -4.79 & 0 & 0 \\ -4.68 & 0 & 0 \\ -5.75 & 0 & 0 \\ -4.79 & 0 & 0 \\ -63.07 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -4.76 & 0 & 0 \\ -4.76 & 0 & 0 \\ -4.63 & 0 & 0 \\ -5.75 & 0 & 0 \\ -4.76 & 0 & 0 \\ -63.07 & 0 & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The input is considered as $u(t) = \begin{bmatrix} T_{0,h} & T_{3,c} \end{bmatrix} = \begin{bmatrix} 77.9°C & 21.2°C \end{bmatrix}$, the fault behavior is given in (5.111). The initial conditions for the nonlinear system (5.54) - (5.55) are $x(0) = [10,\ 60,\ 15,\ 55]^T$, and for the observer (5.3)-(5.5) the initial conditions are $\hat{x}(0) = [0,\ 0,\ 0,\ 0]^T$. The performance of the proposed observer is shown in Figures 5.2 - 5.7. Figures 5.2 to 5.5 show the estimation of the system dynamic states and Figures 5.6 and 5.7 are the algebraic system states and their estimations. Figure 5.8 shows the fault and its estimate.

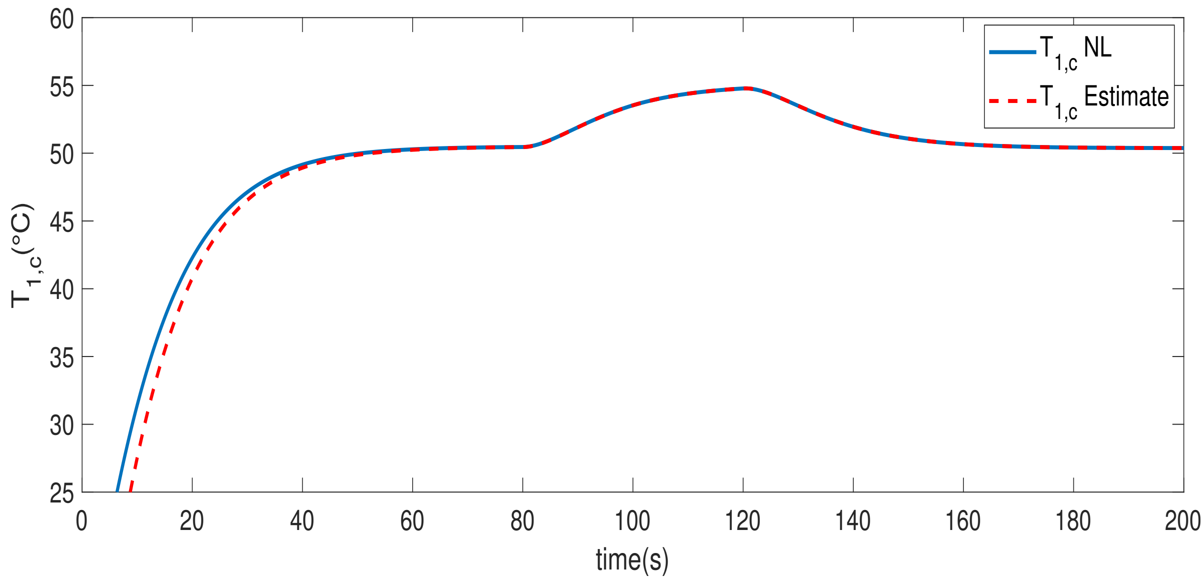

Fig. 5.2. Comparison of state estimate $T_{1,c}(t)$.

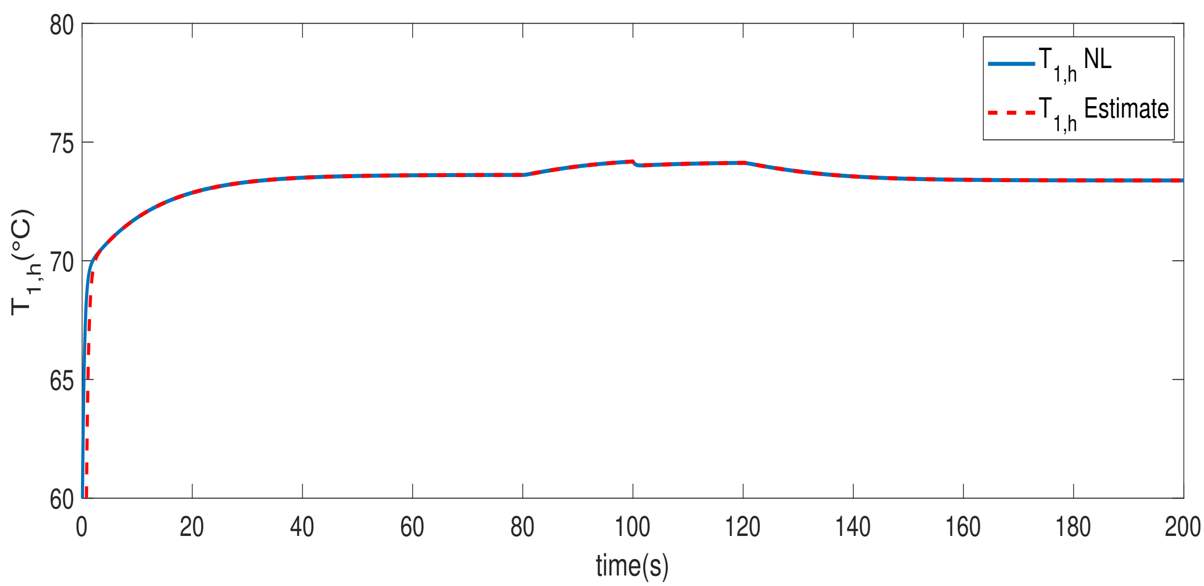

Fig. 5.3. Comparison of state estimate $T_{1,h}(t)$.

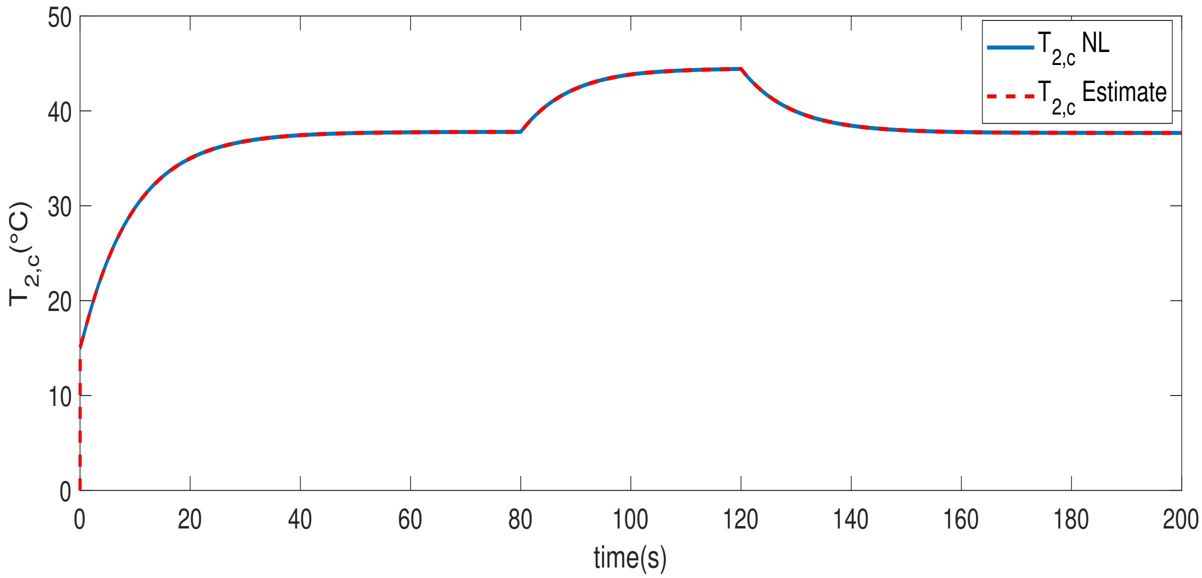

Fig. 5.4. Comparison of state estimate $T_{2,c}(t)$.

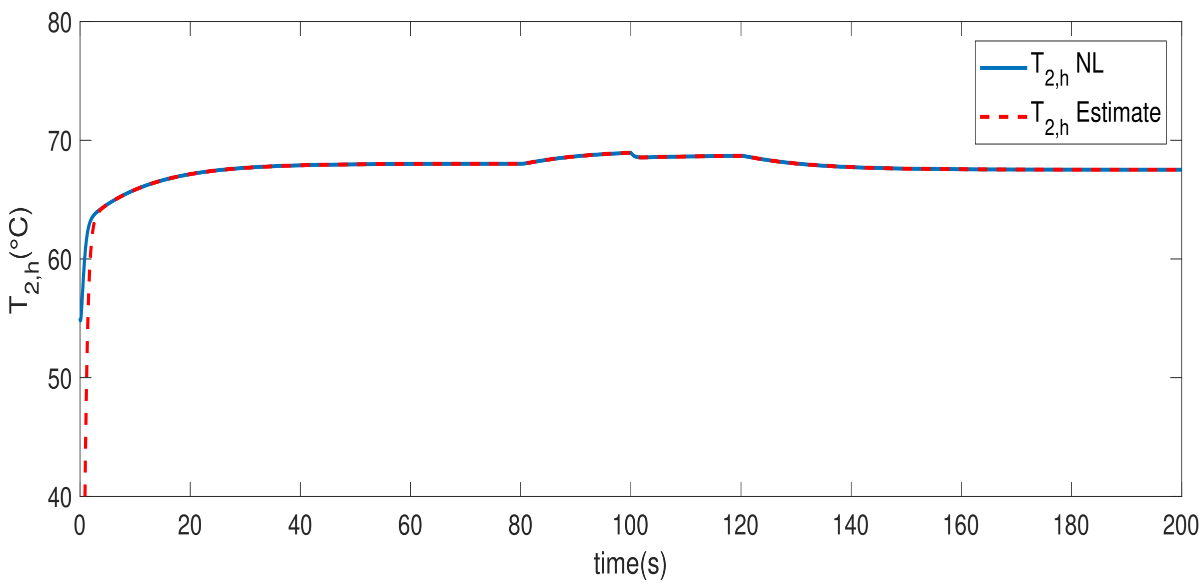

Fig. 5.5. Comparison of state estimate $T_{2,h}(t)$.

The first fault occurs at the time $t = 80s$ and disappears at $t = 120s$. In a real context, the fault can be due to problems in the cold water inlet valve. From the simulation results, we can see that the proposed observer manages to estimate the simultaneous unmeasured states and actuator fault of the system with a reduced convergence time.

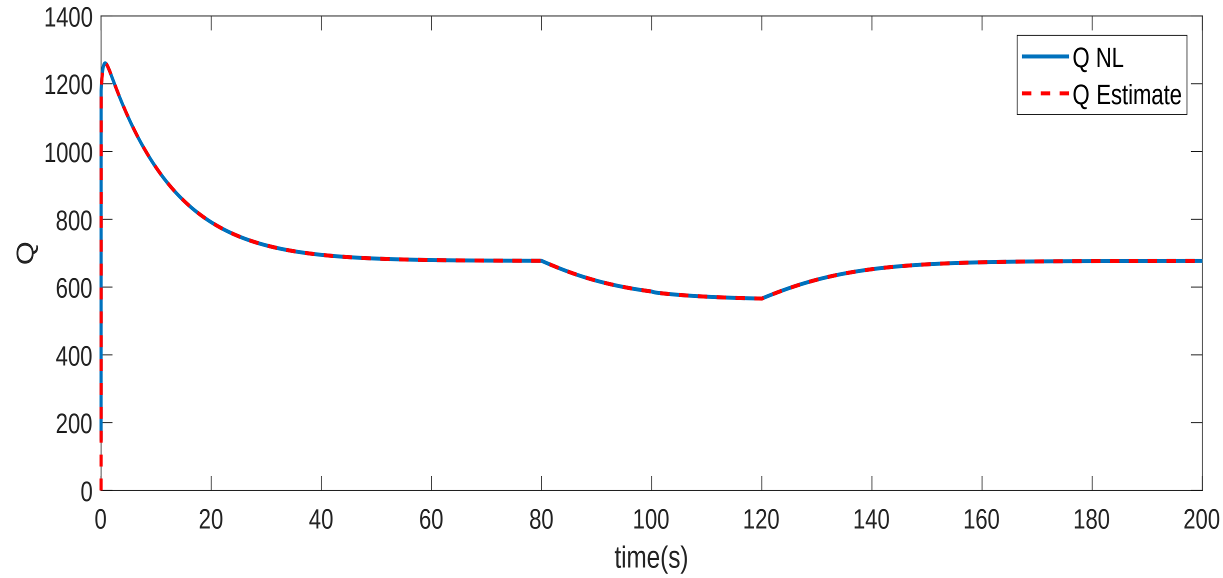

Fig. 5.6. Comparison of state estimate $Q(t)$.

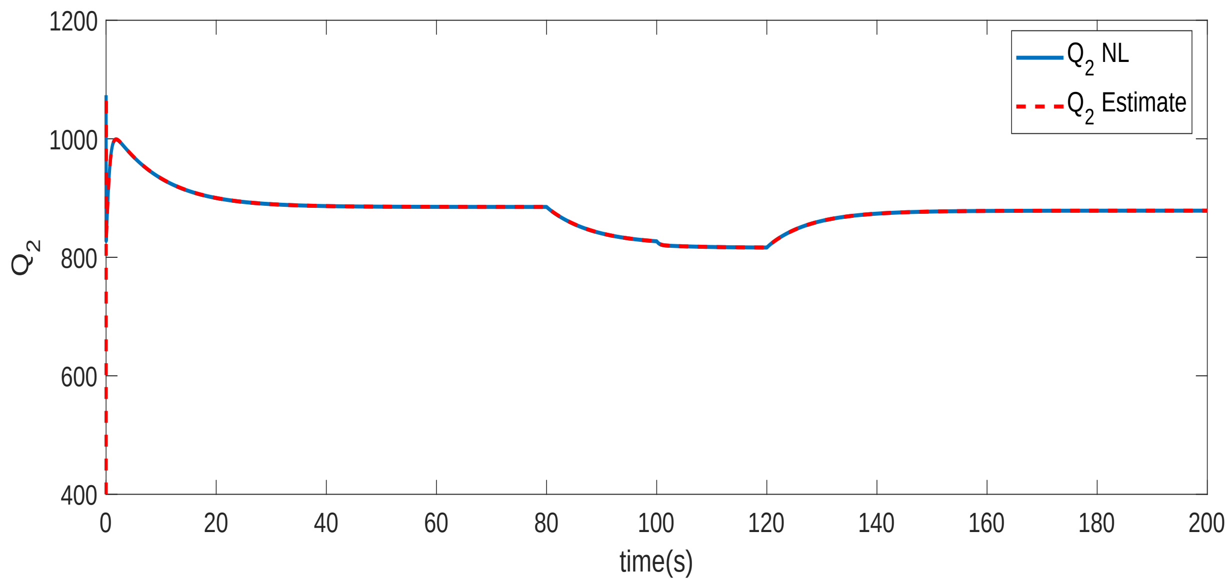

Fig. 5.7. Comparison of state estimate $Q_2(t)$.

To evaluate the performance of the states and fault observer, the integral of the absolute error (IAE) is computed in Table 5.1. It can be concluded that the proposed observer successfully achieves notably accurate state and fault estimation, indicating a good performance for states and fault estimation. **The definition of performance analysis can be found in the appendix A**.

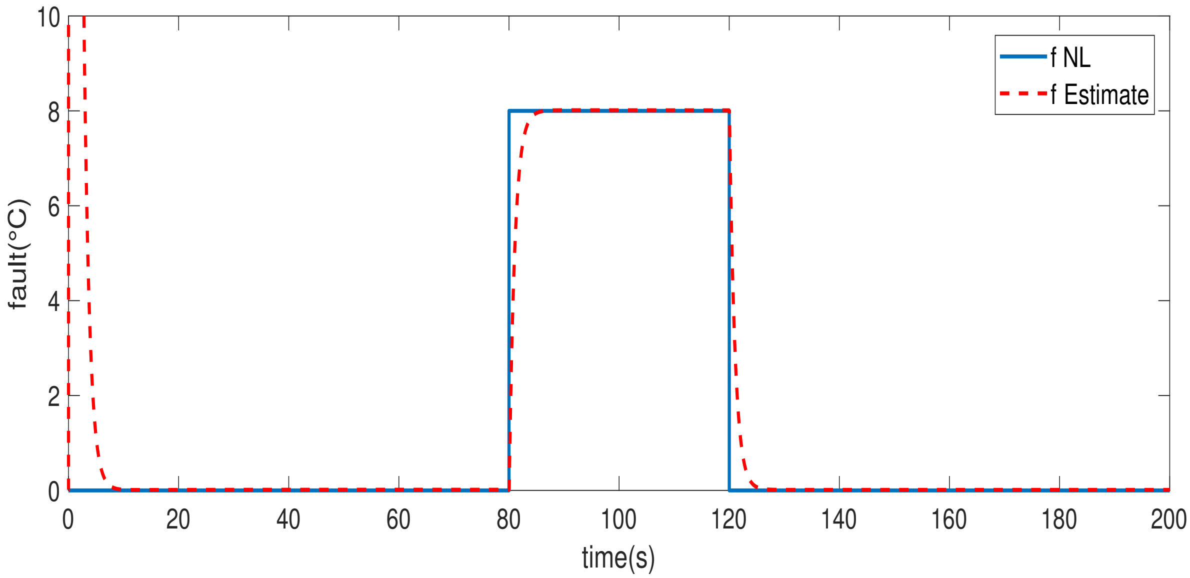

Fig. 5.8. Comparison of fault estimate $f(t)$.

Table 5.1. States and Fault observer performance index

States/Fault	**IAE**
$\hat{x}_1(t) - x_1(t)$	10043
$\hat{x}_2(t) - x_1(t)$	2468
$\hat{x}_3(t) - x_3(t)$	55.7
$\hat{x}_4(t) - x_4(t)$	4652.5
$\hat{x}_5(t) - x_5(t)$	2161.1
$\hat{x}_6(t) - x_6(t)$	1673
$\hat{f}(t) - f(t)$	828.13

5.3 Generalized dynamic learning observer design for S-NLPV systems

5.3.1 Preliminaries

Consider the following D-NLPV system in its polytopic form.

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[A_i x(t) + B_i u(t) + D_i f(t, F_L x)\Big] + G f_a(t), \\ y(t) &= Cx(t), \end{aligned} \tag{5.61}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the known input, $y(t) \in \mathbb{R}^p$ is the measurement output vector and $f_a(t) \in \mathbb{R}^{n_{fa}}$ is the actuator fault vector. Matrix $E \in \mathbb{R}^{n\times n}$ could be singular. $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D_i \in \mathbb{R}^{n\times n_f}$ and $G \in \mathbb{R}^{n\times n_{fa}}$ are real matrices and $f(t, F_L x)$ is a nonlinearity satisfying the Lipschitz constraint $||\Delta_f(t)|| \leq \lambda||F_L(x_1 - x_2)||$ where $\Delta_f(t) = f(t, F_L x_1) - f(t, F_L x_2)$, λ is a known Lipschitz constant and F_L is real matrix of appropriate dimension.
Let $rank(E) = r < n$ and $E^{\perp} \in \mathbb{R}^{s\times n}$ be a full row matrix such that $E^{\perp}E = 0$, in this case $s = n - r$.
Consider $\mu_i(\varrho(t))$ as the membership functions formed with known variant parameters $\varrho(t) \in \mathbb{R}^l$. The membership functions have the following properties:

$$\sum_{i=1}^{k} \mu_i(\varrho(t)) = 1, \quad \mu_i(\varrho(t)) \geq 0, \tag{5.62}$$

for $i = 1, \ldots, k = 2^l$.

Assumption 6. *It is assumed that the system described by equation* (5.61) *is regular (Definition 1), Impulse observable (Definition 2) and Reachable observable (Definition 4).*

Assumption 7. *It is assumed that system* $rank(B_i) = rank\begin{bmatrix} B_i & G \end{bmatrix}$.

Assumption 8. *The actuator fault behavior is assumed to be constant, i.e.* $\dot{f}_a(t) = 0$.

5.3.2 Problem Statement.

Let us consider the following generalized nonlinear observer for system (5.61)

$$\begin{aligned}
\dot{\zeta}(t) &= \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[N_i\zeta(t) + H_i v(t) + F_i y(t) + J_i u(t) + TD_i f(t, F_L\hat{x}) + TG\hat{f}_a(t)\Big], \\
\dot{v}(t) &= \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[S_i\zeta(t) + L_i v(t) + M_i y(t)\Big], && (5.63) \\
\hat{x}(t) &= P\zeta(t) + Qy(t), && (5.64) \\
\hat{f}_a(t) &= \hat{f}_a(t-\tau) + \Phi_{fa}(C\hat{x}(t) - y(t)), && (5.65)
\end{aligned}$$

where τ denotes the sampling time interval, $\zeta(t) \in \mathbb{R}^{q_0}$ represents the state vector of the observer, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector and $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$. The matrices $N_i \in \mathbb{R}^{q_0\times q_0}$, $H_i \in \mathbb{R}^{q_0\times q_1}$, $F_i \in \mathbb{R}^{q_0\times p}$, $S_i \in \mathbb{R}^{q_1\times q_0}$, $L_i \in \mathbb{R}^{q_1\times q_1}$, $M_i \in \mathbb{R}^{q_1\times p}$, $J_i \in \mathbb{R}^{q_0\times m}$, $P \in \mathbb{R}^{n\times q_0}$, $Q \in \mathbb{R}^{n\times p}$, and $T \in \mathbb{R}^{q_0\times n}$ are unknown matrices of appropriate dimensions, which must be determined such that $\hat{x}(t)$ and $\hat{f}_a(t)$ converges asymptotically to $x(t)$ and $f_a(t)$, respectively.

Let a matrix $T \in \mathbb{R}^{q_0\times n}$ to consider the following transformed error

$$\epsilon(t) = \zeta(t) - TEx(t), \tag{5.66}$$

whose derivative is

$$\begin{aligned}\dot{\epsilon}(t) &= \sum_{i=1}^{k}\mu_i(\varrho(t))\Big[N_i\epsilon(t) + N_iTEx(t) + H_iv(t) + F_iCx(t) + \\ &\quad J_iu(t) + TD_if(t, F_L\hat{x}) + TG\hat{f}_a(t) - TA_ix(t) - TB_iu(t) - \\ &\quad TD_if(t, F_Lx) - TGf_a(t)\Big] \qquad (5.67)\\ \dot{\epsilon}(t) &= \sum_{i=1}^{k}\mu_i(\varrho(t))\Big[N_i\epsilon(t) + H_iv(t) + (N_iTE - TA_i + F_iC)x(t) + \\ &\quad (J_i - TB_i)u(t) + TGe_f(t) + TD_i\Delta_f(t)\Big] \qquad (5.68)\end{aligned}$$

where $e_f(t) = \hat{f}_a(t) - f_a(t)$ and $\Delta_f(t) = f(t, F_L\hat{x}) - f(t, F_Lx)$. Equation (5.68) is independent of $x(t)$ and $u(t)$, if the following equations are satisfied:

$$(a) \quad N_iTE + F_iC - TA_i = 0, \qquad (5.69)$$

$$(b) \quad J_i = TB_i. \qquad (5.70)$$

Then equation (5.68) becomes:

$$\dot{\epsilon}(t) = \sum_{i=1}^{k}\mu_i(\varrho(t))\Big[N_i\epsilon(t) + H_iv(t) + TGe_f(t) + TD_i\Delta_f(t)\Big]. \qquad (5.71)$$

By using equation (5.66), equations (5.63) and (5.64) can be written as

$$\dot{v}(t) = \sum_{i=1}^{k}\mu_i(\varrho(t))\big[S_i\epsilon(t) + L_iv(t) + (S_iTE + M_iC)x(t)\big], \qquad (5.72)$$

$$\hat{x}(t) = P\epsilon(t) + (PTE + QC)x(t), \qquad (5.73)$$

Now if the following conditions are satisfied

$$(c) \quad S_iTE + M_iC = 0, \qquad (5.74)$$

$$(d) \quad PTE + QC = I_n, \qquad (5.75)$$

Convergence is ensured if the aforementioned restrictions (a) - (d) are satisfied. However, if these restrictions are not met, the convergence of the observer cannot be guaranteed.

then, equation (5.72) becomes

$$\dot{v}(t) = \sum_{i=1}^{k}\mu_i(\varrho(t)\Big[S_i\epsilon(t) + L_iv(t)\Big], \qquad (5.76)$$

and the state estimation error becomes

$$\hat{x}(t) - x(t) = e(t) = P\epsilon(t). \qquad (5.77)$$

If conditions *(a)–(d)* are satisfied, the following observer error dynamics equation is obtained from (5.71) and (5.76)

$$\dot{\varphi}(t) = \sum_{i=1}^{k}\mu_i(\varrho(t))\Big[\mathbb{A}_i\varphi(t) + \mathbb{B}e_f(t) + \mathbb{C}_i\Delta_f(t)\Big], \qquad (5.78)$$

where $\varphi(t) = \begin{bmatrix} \epsilon(t) \\ v(t) \end{bmatrix}$, $\mathbb{A}_i = \begin{bmatrix} N_i & H_i \\ S_i & L_i \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} TG \\ 0 \end{bmatrix}$ and $\mathbb{C}_i = \begin{bmatrix} TD_i \\ 0 \end{bmatrix}$.

In this case, if $e_f(t) = 0$, $\Delta_f(t) = 0$ and $\begin{bmatrix} N_i & H_i \\ S_i & L_i \end{bmatrix}$ is stable, then $lim_{t\to\infty}\ e(t) = 0$.

Now, the problem of the GLDO (5.63)-(5.65) design is reduced to find matrices N_i, F_i, J_i, H_i, L_i, M_i, S_i, P, Q, T and Φ_{fa}, such that the error dynamic (5.78) is asymptotically stable.

5.3.3 Observer design

This section will be devoted to the parameterization of all the matrices of the observer

Observer parameterization

The parameterization of this observer can be obtained in a manner similar to that in the section. 4.2.2.

Now, by using the value of matrices N_i, S_i and T given by (4.25), (4.31) and (4.20), respectively, the observer error dynamics (5.78) can be written as

$$\dot{\varphi}(t) \ = \ \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[\bar{\mathbb{A}}_i\varphi(t) + \mathbb{B}e_f(t) + \mathbb{C}_i\Delta_f(t)\Big], \tag{5.79}$$

where $\bar{\mathbb{A}}_i = \mathbb{A}_{1i} - \mathbb{Y}_i\mathbb{A}_2$, $\mathbb{A}_{1i} = \begin{bmatrix} N_{1i} - Z_1N_{2i} & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} T_1G - Z_1T_2G \\ 0 \end{bmatrix}$, $\mathbb{C}_i = \begin{bmatrix} T_1D_i - Z_1T_2D_i \\ 0 \end{bmatrix}$ and $\mathbb{Y}_i = \begin{bmatrix} Y_{1i} & H_i \\ Y_{2i} & L_i, \end{bmatrix}$,
and from (5.77) we have

$$e(t) \ = \ \mathbb{P}\varphi(t), \tag{5.80}$$

where $\mathbb{P} = \begin{bmatrix} P_1 & 0 \end{bmatrix}$ and $Y_3 = 0$ for simplicity, without loss of generality. Considering equation (5.65), $\tilde{f}_a(t) = f_a(t-\tau) - f_a(t)$, $e_f(t-\tau) = \hat{f}_a(t-\tau) - f_a(t-\tau)$ and the definition of $e_f(t)$, we have

$$\begin{aligned} e_f(t) &= \hat{f}_a(t-\tau) + \Phi_{fa}C(\hat{x}(t) - x(t)) - f_a(t) \\ e_f(t) &= e_f(t-\tau) + \Phi_{fa}Ce(t) + \tilde{f}_a(t) \end{aligned} \tag{5.81}$$

From Assumption 8 it is assumed that $\tilde{f}_a(t)$ can be made zero, so then, estimation error (5.81) can be expressed as

$$e_f(t) = \ e_f(t-\tau) + \mathbb{K}\varphi(t). \tag{5.82}$$

where $\mathbb{K} = \begin{bmatrix} \Phi_{fa}CP_1 & 0 \end{bmatrix}$.

The observer design is obtained from the determination of matrices Z_1 and $\mathbb{Y}_i$ such that system (5.79) is asymptotically stable.

5.3.4 Stability analysis of the observer

This section is devoted to the stability analysis of equation (5.79).

Theorem 3. *Under Assumption 6, there exist two parameter matrices Z_1 and $\mathbb{Y}_i$ such that system (5.79) is asymptotically stable if there exists a positive definite matrix $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} > 0$ such that the following LMIs are satisfied*

$$\mathcal{C}^{T\perp} \begin{bmatrix} \Pi_{1i} + \Pi_2 & \Pi_3 \\ (*) & -\gamma I_{nf} \end{bmatrix} \mathcal{C}^{T\perp T} < 0, \tag{5.83}$$

$$\gamma > 0, \tag{5.84}$$

$$\begin{bmatrix} -\eta I & \mathbb{B}^T X + \mathbb{K} \\ (*) & -I \end{bmatrix} < 0 \tag{5.85}$$

where

$$\begin{aligned} \mathcal{C}^{T\perp} = \begin{bmatrix} \mathbb{A}_2^{T\perp} & 0 \\ 0 & I \end{bmatrix}, \Pi_2 = \gamma\lambda^2 \begin{bmatrix} {P_1}^T F_L^T F_L P_1 & 0 \\ 0 & 0, \end{bmatrix}, \Pi_3 = \begin{bmatrix} X_1 T_1 D_i - X_z T_2 D_i \\ 0 \end{bmatrix} \\ \Pi_{1i} = \begin{bmatrix} X_1 N_{1i} + N_{1i}^T X_1 - X_z N_{2i} - N_{2i}^T X_z^T & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{5.86}$$

in this case $\eta > 0$, matrix $Z_1 = X_1^{-1} X_z$ and parameter matrix $\varPhi_i^T$ is obtained as follows

$$\varPhi_i^T = {\mathcal{B}_r}^+ \mathcal{K}_i {\mathcal{C}_l}^+ + \mathcal{Z} - {\mathcal{B}_r}^+ \mathcal{B}_r \mathcal{Z} \mathcal{C}_l {\mathcal{C}_l}^+, \tag{5.87}$$

where

$$\mathbb{Y}_i = (\varPhi_i X_1^{-1})^T, \tag{5.88}$$

$$\mathcal{K}_i = -\mathcal{R}^{-1} {\mathcal{B}_l}^T \mathcal{V}_i {\mathcal{C}_r}^T (\mathcal{C}_r \mathcal{V}_i {\mathcal{C}_r}^T)^{-1} + \mathcal{S}_i^{\frac{1}{2}} \mathcal{L} (\mathcal{C}_r \mathcal{V}_i {\mathcal{C}_r}^T)^{-\frac{1}{2}}, \tag{5.89}$$

$$\mathcal{S}_i = \mathcal{R}^{-1} - \mathcal{R}^{-1} \mathcal{B}_l^T [\mathcal{V}_i - \mathcal{V}_i \mathcal{C}_r^T (\mathcal{C}_r \mathcal{V}_i {\mathcal{C}_r}^T)^{-1} \mathcal{C}_r \mathcal{V}_i] \mathcal{B}_l \mathcal{R}^{-1}, \tag{5.90}$$

$$\mathcal{V}_i = (\mathcal{B}_r \mathcal{R}^{-1} {\mathcal{B}_l}^T - \mathcal{D}_i)^{-1} > 0, \tag{5.91}$$

where matrices $\mathcal{Z}$, $\mathcal{L}$, $\mathcal{R}$ are arbitrary matrices such that $||\mathcal{L}|| < 1$ and $\mathcal{R} > 0$, with

$$\mathcal{D}_i = \begin{bmatrix} \Pi_{1i} + \Pi_2 & \Pi_3 \\ (*) & -\gamma I_{nf} \end{bmatrix},$$

$\mathcal{B} = \begin{bmatrix} -I \\ 0 \end{bmatrix}$ *and* $\mathcal{C} = \begin{bmatrix} \mathbb{A}_2 & 0 \end{bmatrix}$*, such that there exist matrices $\mathcal{B}_l$, $\mathcal{B}_r$, $\mathcal{C}_l$ and $\mathcal{C}_r$, are such that $\mathcal{B} = \mathcal{B}_l \mathcal{B}_r$ and $\mathcal{C} = \mathcal{C}_l \mathcal{C}_r$, respectively.*

Proof. Consider the following Lyapunov candidate function

$$V(\varphi(t)) = \varphi^T(t)X\varphi(t) + \int_{t-\tau}^{t} e_f^T(t)e_f(t)dt \tag{5.92}$$

such that

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} > 0, \quad X_1 = X_1^T. \tag{5.93}$$

The derivative of (5.92) along the trajectory of (5.79) gives

$$\dot{V}(\varphi(t)) \leq \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[\varphi^T(t)X\dot{\varphi}(t) + \dot{\varphi}^T(t)X\varphi(t) + e_f^T(t)e_f(t) - e_f^T(t-\tau)e_f(t-\tau)\Big], \tag{5.94}$$

replacing (5.79) and considering (5.82) in equation (5.94), we have

$$\begin{aligned} &\dot{V}(\varphi(t)) \leq \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[\varphi^T(t)(\bar{\mathbb{A}}_i^T X + X^T\bar{\mathbb{A}}_i)\varphi(t) + \\ &\varphi^T(t)X^T\mathbb{C}_i\Delta_f(t) + \Delta_f^T(t)\mathbb{C}_i^T X\varphi(t) + \\ &2\varphi^T(t)[X\mathbb{B} + \mathbb{K}^T]e_f(t-\tau) + \varphi^T(t)[2X\mathbb{B} + \mathbb{K}^T]\mathbb{K}\varphi(t)\Big], \end{aligned} \tag{5.95}$$

taking into account the following restriction

$$\mathbb{B}^T X = -\mathbb{K} \tag{5.96}$$

we can obtain the following equivalence

$$\varphi^T(t)[2X\mathbb{B} + \mathbb{K}^T]\mathbb{K}\varphi(t) = -\varphi(t)\mathbb{K}^T\mathbb{K}\varphi(t) < 0 \tag{5.97}$$

It is important to note that Equation (5.97) is formed by a quadratic term and by containing the negative sign, the condition of being defined negative will be fulfilled as long as $\varphi(t) \neq 0$.

Now, Equation (5.95) can be written as

$$\dot{V}(\varphi(t)) \leq \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[\varphi^T(t)(\bar{\mathbb{A}}_i^T X + X^T\bar{\mathbb{A}}_i)\varphi(t) + \varphi^T(t)X^T\mathbb{C}_i\Delta_f(t) + \Delta_f^T(t)\mathbb{C}_i^T X\varphi(t)\Big], \tag{5.98}$$

By using Lemma 3 we can obtain the following inequality

$$\Delta_f^T(t)\mathbb{C}_i^T X\varphi(t) + \varphi^T(t)X^T\mathbb{C}_i\Delta_f(t) \leq \gamma\Delta_f^T(t)\Delta_f(t) + \frac{1}{\gamma}\varphi^T(t)X^T\mathbb{C}_i\mathbb{C}_i^T X\varphi(t),$$

and from the Lipschitz condition, we have

$$\Delta_f^T\Delta_f(t) \leq \gamma\lambda^2\varphi^T(t)\mathbb{P}^T F_L^T F_L\mathbb{P}\varphi(t), \tag{5.99}$$

by inserting (5.99) and (5.99) into (5.98), we obtain

$$\dot{V}(\varphi(t)) \leq \sum_{i=1}^{k} \mu_i(\varrho(t))\left[\varphi^T(t)\left(\bar{\mathbb{A}}_i^T X + X^T\bar{\mathbb{A}}_i + \frac{1}{\gamma}X^T\mathbb{C}_i\mathbb{C}_i^T X + \gamma\lambda^2\mathbb{P}^T F_L^T F_L\mathbb{P}\right)\varphi(t)\right]. \tag{5.100}$$

Now, if the following LMI is satified then $\dot{V}(\varphi(t)) < 0$.

$$\bar{\mathbb{A}}_i^T X + X^T \bar{\mathbb{A}}_i + \frac{1}{\gamma} X^T \mathbb{C}_i \mathbb{C}_i^T X + \gamma \lambda^2 \mathbb{P}^T F_L^T F_L \mathbb{P} < 0, \tag{5.101}$$

By using the Schur complement (Lemma 1) and replacing matrix $\bar{\mathbb{A}}_i$ in inequality (5.101), we obtain

$$\left[\begin{array}{c|c} X(\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2) + (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)^T X + \gamma \lambda^2 \mathbb{P}^T F_L^T F_L \mathbb{P} & X\mathbb{C}_i \\ \hline (*) & -\gamma I_{nf} \end{array}\right] < 0, \tag{5.102}$$

Now, by inserting the values of matrices $\mathbb{A}_{1i}$, $\mathbb{P}$ and X we obtain

$$\begin{bmatrix} \Pi_{1i} + \Pi_2 - \mathbb{A}_2^T \Phi_i - \Phi_i^T \mathbb{A}_2 & \Pi_3 \\ (*) & -\gamma I_{nf} \end{bmatrix} < 0, \tag{5.103}$$

where $\Phi_i = \mathbb{Y}_i^T X$ and $X_z = X_1 Z_1$. Π_{1i}, Π_2 and Π_3 are defined in (5.86). Inequality (5.103) can also be rewritten as

$$\mathcal{B}\Phi_i^T \mathcal{C} + (\mathcal{B}\Phi_i^T \mathcal{C})^T + \mathcal{D}_i < 0, \tag{5.104}$$

where

$$\mathcal{D}_i = \begin{bmatrix} \Pi_{1i} + \Pi_2 & \Pi_3 \\ (*) & -\gamma I_{nf} \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} -I \\ 0 \end{bmatrix} \text{ and } \mathcal{C} = \begin{bmatrix} \mathbb{A}_2 & 0 \end{bmatrix}.$$

According to Lemma 2, inequality (5.104) is satisfied if and only if the following inequalities verified

$$\mathcal{C}^{T\perp} \mathcal{D}_i \mathcal{C}^{T\perp T} < 0, \tag{5.105}$$

$$\mathcal{B}^{\perp} \mathcal{D}_i \mathcal{B}^{\perp T} < 0, \tag{5.106}$$

the inequalities (5.105) and (5.106) are equivalent to (5.107) and (5.108) respectively

$$\mathcal{C}^{T\perp} \begin{bmatrix} \Pi_{1i} + \Pi_2 & \Pi_3 \\ (*) & -\gamma I_{nf} \end{bmatrix} \mathcal{C}^{T\perp T} < 0, \tag{5.107}$$

$$\gamma > 0. \tag{5.108}$$

with $\mathcal{B}^{\perp} = \begin{bmatrix} 0 & I \end{bmatrix}$ and $\mathcal{C}^{T\perp} = \begin{bmatrix} \mathbb{A}_2^{T\perp} & 0 \\ 0 & I \end{bmatrix}$. If condition (5.96) is satisfied, conditions (5.107) and (5.108) can be solved using a standard tool for linear matrix inequalities (LMIs). However, condition (5.96) is a matrix equality to solve it, it can be rewritten as (Jia et al., 2016)

$$(\mathbb{B}^T X + \mathbb{K})(\mathbb{B}^T X + \mathbb{K})^T < \eta^2 I \tag{5.109}$$

where η is a positive scalar. By using the Schur complement lemma (Boyd et al., 1994), equation (5.109) can be written as

$$\begin{bmatrix} -\eta^2 I & \mathbb{B}^T X + \mathbb{K} \\ (*) & -I \end{bmatrix} < 0 \tag{5.110}$$

The design problem can be simplified by considering a scalar $\eta > 0$ and $\gamma > 0$ to solve the inequalities (5.83), (5.84), and (5.85), resulting in a positive definite matrix X. The parameter matrix $\mathbb{Y}_i$ can be obtained as (5.88), which completes the proof of the theorem. ■

□

5.3.5 Simulation applied to heat exchanger

This section is devoted to the performance analysis of the observer (5.63)-(5.65) applied to the heat exchanger model similar to that presented in Section 5.2.5. First, we can confirm that Assumptions 3, 4, and 5 are satisfied. Then, we can proceed to apply a procedure similar to the algorithm outlined in Section 5.2.4.

The nonlinearity $f(t, F_L x)$ is considered as a Lipschitz function. The input signal is considered constant $u(t) = \begin{bmatrix} 77.9°C & 21.2°C \end{bmatrix}$. The fault signal is defined as

$$f_a(t) = \begin{cases} 8°C & 80s \leq t \leq 120s \\ 0°C & \text{otherwise (s).} \end{cases} \tag{5.111}$$

The Lipschitz constant is selected as $\lambda = 100$ and $\tau = 0.00001$. The initial conditions for the nonlinear system are $x(0) = [10,\ 60,\ 15,\ 55]^T$, and for the observer (5.63) - (5.65) $\hat{x}(0) = [0,\ 0,\ 0,\ 0]^T$. The perfomance of the proposed observer is shown in Figures 5.9 - 5.10. The fault occurs at the time $t = 80s$. In a real context, the fault can be due to problems in the cold water inlet valve. From the simulation results shown in Figure 5.9, it can be observed that the proposed observer successfully estimates the simultaneous unmeasured states of the system, achieving a notable reduction in convergence time. The estimate fault manages to reduce the convergence time due to the learning part as seen in Figure 5.10.

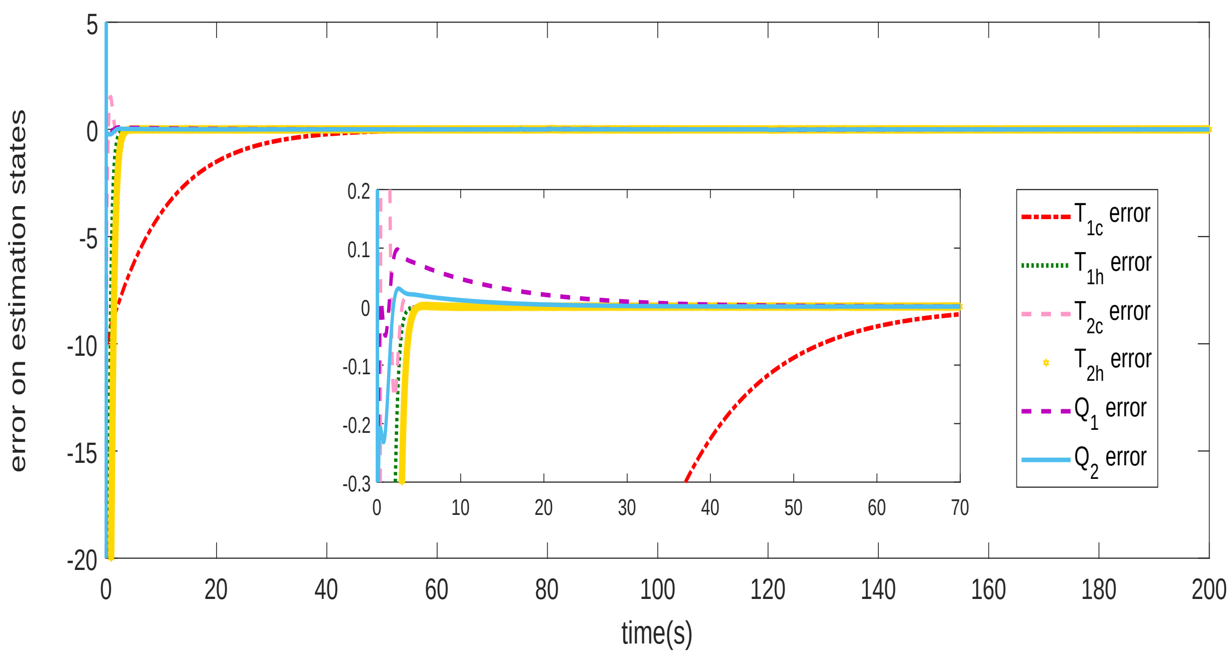

Fig. 5.9. Convergence of the states estimation errors.

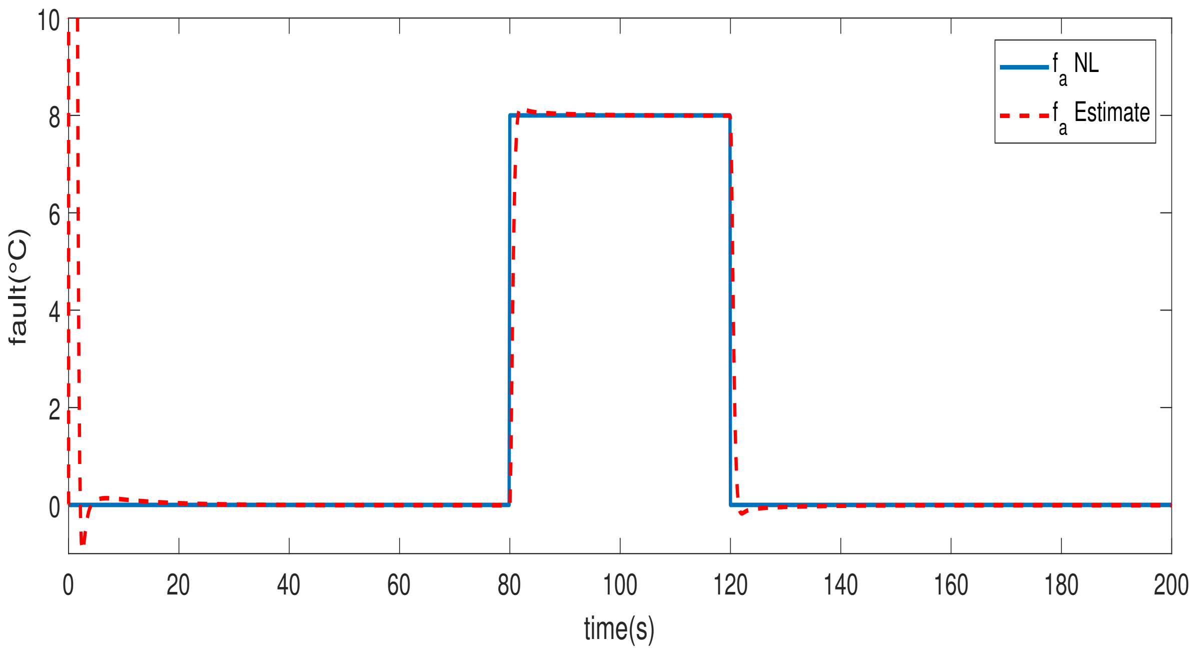

Fig. 5.10. Fault estimation.

5.3.6 Conclusions

In this chapter, a GDO structure is presented to perform simultaneous estimation of state variables and actuator faults in algebro-differential nonlinear parameter varying systems. A second design for reducing the convergence time for fault estimation is presented. The conditions for the existence of this observer design is given in the form of a set of LMIs. This methodology can be applied to standard LTI formulations, considering them as particular cases. In order to illustrate the observer performances, a heat exchanger with two countercurrent cells was considered.

Chapter 6

Generalized dynamic adaptive observers for parameter estimation

6.1 Introduction

This chapter presents an adaptive observer design for simultaneous estimation of system parameters and state variables for a class of linear descriptor systems. In Section 6.2, a case where variable parameters are present in the system is considered. The observer design is obtained in terms of a set of linear matrix inequalities (LMI), and the conditions of existence and stability are given. Academic examples illustrate the efficiency of the proposed approaches.

6.2 Generalized dynamic LPV adaptive observer for parameter estimation

6.2.1 Preliminaries

Let us consider the following linear algebro-differential system:

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[A_i x(t) + B_i u(t)\Big] + \psi(t)\theta(t), \\ y(t) &= Cx(t), \end{aligned} \tag{6.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the known input and $y(t) \in \mathbb{R}^p$ is the measurement output vector and $\theta(t) \in \mathbb{R}^l$ is the unknown parameter vector assumed to be constant. Matrix $E \in \mathbb{R}^{n\times n}$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are known constant matrices. $\psi(t) \in \mathbb{R}^{m\times l}$ is a matrix of known signals, and it is assumed to be piecewise differentiable. Both $\psi(t)$ and its derivative $\dot{\psi}(t)$ are uniformly bounded in time. Assume that: $rank(E) = r < n$, and without loss of generality: $rank(C) = p$.

The problem addressed in this paper involves the simultaneous estimation of $x(t)$ and $\theta(t)$ using measured and known signals $u(t)$, $y(t)$, and $\psi(t)$. Now since $rank(E) < n$, there always exists a full row rank matrix $E^{\perp}$ such that:

$$E^{\perp}E = 0. \tag{6.2}$$

Following (6.2), system (6.1) is equivalent to:

$$\sum_{i=1}^{k}\mu_i(\varrho(t))\Big[\begin{bmatrix}-E^{\perp}B_iu(t)\\ y(t)\end{bmatrix}\Big] = \sum_{i=1}^{k}\mu_i(\varrho(t))\Big[\begin{bmatrix}E^{\perp}A_i\\ C\end{bmatrix}x(t)\Big] + \begin{bmatrix}E^{\perp}\psi\\ 0\end{bmatrix}\theta(t) \tag{6.3}$$

Our aim is to design an adaptive observer of the form:

$$\begin{aligned}
\dot{\zeta}(t) &= \sum_{i=1}^{k}\mu_i(\varrho(t))\Big[N_i\zeta + H_iv + F_i\left(\begin{bmatrix}-E^{\perp}B_iu(t)\\ y(t)\end{bmatrix} - \begin{bmatrix}E^{\perp}\psi\\ 0\end{bmatrix}\hat{\theta}(t)\right) + \\
&\quad J_iu(t)\Big] + T\psi\hat{\theta}(t) + \Upsilon_1\dot{\hat{\theta}}(t), && (6.4)\\
\dot{v}(t) &= \sum_{i=1}^{k}\mu_i(\varrho(t))\Big[S_i\zeta + L_iv + M_i\left(\begin{bmatrix}-E^{\perp}B_iu(t)\\ y(t)\end{bmatrix} - \begin{bmatrix}E^{\perp}\psi\\ 0\end{bmatrix}\hat{\theta}(t)\right)\Big], && (6.5)\\
\hat{x} &= \sum_{i=1}^{k}\mu_i(\varrho(t))\Big[P\zeta + Q\left(\begin{bmatrix}-E^{\perp}B_iu(t)\\ y(t)\end{bmatrix} - \begin{bmatrix}E^{\perp}\psi\\ 0\end{bmatrix}\hat{\theta}(t)\right)\Big], && (6.6)\\
\dot{\hat{\theta}} &= \Gamma\beta^T(t)C^T\Lambda(y(t) - C\hat{x}(t)), && (6.7)
\end{aligned}$$

where $\hat{x}(t)$ and $\hat{\theta}(t)$ are the estimates of $x(t)$ and $\theta(t)$, respectively. Matrices $N_i \in \mathbb{R}^{q_0\times q_0}$, $H_i \in \mathbb{R}^{q_0\times q_1}$, $F_i \in \mathbb{R}^{q_0\times p}$, $S_i \in \mathbb{R}^{q_1\times q_0}$, $L_i \in \mathbb{R}^{q_1\times q_1}$, $M_i \in \mathbb{R}^{q_1\times p}$, $J_i \in \mathbb{R}^{q_0\times m}$, $P \in \mathbb{R}^{n\times q_0}$, $Q \in \mathbb{R}^{n\times p}$, and $T \in \mathbb{R}^{q_0\times n}$, $\Gamma(t)$ and $\beta(t)$ are unknown matrices of appropriate dimensions, which must be determined such that $\hat{x}(t)$ converges asymptotically to $x(t)$ and $\hat{\theta}(t)$ converges to $\theta(t)$, respectively. $\Gamma \in \mathbb{R}^l$ and $\Lambda \in \mathbb{R}^l$ are symmetric positive matrices which are used to adjust the evolution rate of $\hat{\theta}(t)$.

6.2.2 Problem statement

Let us consider the two estimation errors

$$\begin{aligned}
e_x(t) &= x(t) - \hat{x}(t).\\
e_\theta(t) &= \dot{\theta}(t) - \dot{\hat{\theta}}(t).
\end{aligned} \tag{6.8}$$

Now, we make the following two assumptions which are needed for the rest of the paper

Assumption 9. *It is assumed that the system described by equation* (6.1) *is regular (Definition 1), Impulse observable (Definition 2) and Reachable observable (Definition 4).*

Assumption 10. *Let $\beta(t) \in \mathbb{R}^{n\times l}$ be a matrix of signals generated by the stable ODE system of the form:*

$$\dot{\beta}(t) = \sum_{i=1}^{k}\mu_i(\varrho(t))\Big[\mathbb{A}_i\beta(t) + \mathcal{X}_i(t)\Big]. \tag{6.9}$$

Assuming that input signal $\mathcal{X}$ are persistently exciting, so there exist two positive constants, δ and t_1 and some bounded symmetric positive definitive matrix $\Lambda \in \mathbb{R}^{q\times q}$ such that for all t the following inequality holds:

$$\int_t^{t-t_1} \beta^T(\tau) C^T \Lambda C \beta(\tau) d\tau \geq \delta I. \tag{6.10}$$

Remark 1. *Condition (6.10) is typically required for parameter identification.*
The following theorem gives the conditions for the existence of the observer:

Theorem 4. *Let $\Gamma \in \mathbb{R}^{l\times l}$ be any symmetric positive definite matrix. Under Assumptions 9 and 10, and for a constant $\theta(t)$, the adaptive observer (6.4) - (6.7), is a global exponential adaptive observer for the descriptor system (6.1) if there exist a matrix T such that:*

$$\begin{aligned} J_i &= TB_i, && (6.11)\\ N_iTE + F_i \begin{bmatrix} E^{\perp}A_i \\ C \end{bmatrix} &= TA_i, && (6.12)\\ S_iTE + M_i \begin{bmatrix} E^{\perp}A_i \\ C \end{bmatrix} &= 0, && (6.13)\\ PTE + Q \begin{bmatrix} E^{\perp}A_i \\ C \end{bmatrix} &= I_n, && (6.14) \end{aligned}$$

and $\mathbb{A}_i = \begin{bmatrix} N_i & -H_i \\ -S_i & L_i \end{bmatrix}$ are a stable matrices.
The matrix of signals $\Upsilon(t) = \begin{bmatrix} \Upsilon_1(t) \\ 0 \end{bmatrix}$ is obtained by linearly filtering $\psi(t)$ through:

$$\dot{\Upsilon}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t)) \Big[\mathbb{A}_i \Upsilon(t) + \mathbb{D}_i \psi(t) \Big], \tag{6.15}$$

where $\mathbb{D}_i = \begin{bmatrix} T - F_i \begin{bmatrix} E^{\perp} \\ 0 \end{bmatrix} \\ M_i \begin{bmatrix} E^{\perp} \\ 0 \end{bmatrix} \end{bmatrix}$.

The matrix $\beta(t)$ is given from equation (6.9) such that:

$$\mathcal{X}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t)) \left[\mathbb{A}_i Q \begin{bmatrix} E^{\perp} \\ 0 \end{bmatrix} + \mathbb{D}_i \right] \psi(t) - Q \begin{bmatrix} E^{\perp} \\ 0 \end{bmatrix} \dot{\psi}(t) \tag{6.16}$$

or by:

$$\beta(t) = \mathbb{P}\Upsilon(t) - Q \begin{bmatrix} E^{\perp} \\ 0 \end{bmatrix} \psi(t), \tag{6.17}$$

with $\mathbb{P} = \begin{bmatrix} P & 0 \end{bmatrix}$.

Proof. Let us define the transformation error $\epsilon(t)$ as:

$$\epsilon(t) = TEx(t) - \zeta(t) \tag{6.18}$$

its derivative is given by

$$\begin{aligned} \dot{\epsilon}(t) \;\; = \;\; & \sum_{i=1}^{k} \mu_i(\varrho(t)) \Big[N_i \epsilon(t) - H_i v(t) \\ & - \left(N_i TE - TA_i + F_i \begin{bmatrix} E^{\perp} A_i \\ C \end{bmatrix} \right) x(t) + (TB_i - J_i) u(t) \\ & + \left(T - F \begin{bmatrix} E^{\perp} \\ 0 \end{bmatrix} \right) \psi(t) e_\theta(t) \Big] - \Upsilon_1 \dot{\hat{\theta}}(t), \end{aligned} \tag{6.19}$$

Using (6.11) and (6.12), we can obtain:

$$\dot{\epsilon}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t)) \Big[N_i \epsilon(t) - H_i v(t) + \left(T - F_i \begin{bmatrix} E^{\perp} \\ 0 \end{bmatrix} \right) \psi(t) e_\theta + \Upsilon_1 \dot{e}_\theta \Big], \tag{6.20}$$

Now, by utilizing (6.18), equation (6.5) can be expressed as:

$$\dot{v}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t)) \Big[- S_i \epsilon(t) + L_i v(t) + (S_i TE + M_i \begin{bmatrix} E^{\perp} A_i \\ C \end{bmatrix}) x(t) + M_i \begin{bmatrix} E^{\perp} \\ 0 \end{bmatrix} \psi(t) e_\theta(t) \Big], \tag{6.21}$$

Using (6.13) we have

$$\dot{v}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t)) \Big[- S_i \epsilon(t) + L_i v(t) + M_i \begin{bmatrix} E^{\perp} \\ 0 \end{bmatrix} \psi(t) e_\theta(t) \Big], \tag{6.22}$$

Considering equations (6.19) and (6.21), the following observer error dynamics equation is obtained

$$\dot{\varphi}(t) \;\; = \;\; \sum_{i=1}^{k} \mu_i(\varrho(t)) \Big[\mathbb{A}_i \varphi(t) + \mathbb{D}_i \psi(t) e_\theta + \Upsilon(t) \dot{e}_\theta(t) \Big], \tag{6.23}$$

where $\varphi(t) = \begin{bmatrix} \epsilon(t) \\ v(t) \end{bmatrix}$.

The key step of the proof is to define the following linear combination of $\varphi(t)$ and $e_\theta(t)$:

$$\eta(t) = \varphi(t) - \Upsilon(t) e_\theta(t), \tag{6.24}$$

then we have:

$$\dot{\eta}(t) = \dot{\varphi}(t) - \dot{\Upsilon}(t)e_\theta(t) - \Upsilon(t)\dot{e}_\theta(t). \tag{6.25}$$

By substituting equation (6.23) into equation (6.25), we obtain:

$$\dot{\eta}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[\mathbb{A}_i\eta(t) + \Big(\mathbb{D}_i\psi(t) + \mathbb{A}_i\Upsilon(t) - \dot{\Upsilon}(t)\Big)\, e_\theta(t)\Big] \tag{6.26}$$

From linear equation (6.15), equation (6.26) its reduced to its homogeneous part:

$$\dot{\eta}(t) = \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[\mathbb{A}_i\eta(t)\Big], \tag{6.27}$$

Matrices $\mathbb{A}_i$ must be stability matrices to ensure convergence of $\eta(t)$ to 0.

Now and since the parameter vector $\theta(t)$ is constant, we have:

$$\dot{\theta}(t) = 0, \tag{6.28}$$

and

$$\begin{aligned} \dot{e}_\theta(t) &= -\dot{\hat{\theta}}(t), \\ &= -\Gamma\beta^T(t)C^T\Lambda(y(t) - \hat{y}(t)), \\ &= -\Gamma\beta^T(t)C^T\Lambda C e_x(t). \end{aligned} \tag{6.29}$$

On the other hand, by taking (6.14), we can easily prove that

$$e_x(t) = \mathbb{P}\varphi(t) - Q\begin{bmatrix} E^\perp \\ 0 \end{bmatrix}\psi e_\theta(t), \tag{6.30}$$

where $\mathbb{P} = \begin{bmatrix} P & 0 \end{bmatrix}$.

Then, by replacing $\varphi(t)$ from (6.24), and using (6.30) in (6.29), we obtain:

$$\dot{e}_\theta = -\Gamma\beta^T(t)C^T\Lambda C\left[\mathbb{P}\eta(t) + [\mathbb{P}\Upsilon(t) - Q\begin{bmatrix} E^\perp\psi(t) \\ 0 \end{bmatrix}]e_\theta(t)\right], \tag{6.31}$$

by choosing $\beta(t)$ as in (6.17), equation (6.31) becomes:

$$\dot{e}_\theta = -\Gamma\beta^T(t)C^T\Lambda C\left[\mathbb{P}\eta(t) + \beta(t)e_\theta(t)\right], \tag{6.32}$$

Looking to the homogeneous part of system (6.32) leads to:

$$\dot{e}_\theta = -\Gamma\beta^T(t)C^T\Lambda C\beta(t)e_\theta(t). \tag{6.33}$$

As $\psi(t)$ and $\dot{\psi}(t)$ are bounded, $\beta(t)$ generated from the exponentially stable system (6.9) is also bounded. From the persistent excitation condition (6.10), and since Γ and Λ are positive, the global exponential stability of system (6.33) can be easily proved. From the exponential convergence of $\eta(t)$ and of system (6.33), we obtain the global and exponential convergence to 0 of $e_\theta(t)$ generated from system (6.32) (see Zhang (2002), Alma and Darouach (2014) and Alma et al. (2018) for more details).

Now, from $\eta(t) \to 0$, $e_\theta(t) \to 0$ and the fact that $\Upsilon(t)$ is bounded, we conclude that $\varphi(t) = \sum_{i=1}^{k} \mu_i(\varrho(t))\Big[\eta_i(t) + \Upsilon(t)e_\theta(t)\Big]$ converges to 0, with global and exponential convergence.

Finally, from the convergence of $e_\theta(t)$ and $\varphi(t)$ to 0, and from (6.30), the global and exponential convergence of $e_x(t) \to 0$ is guaranteed. □

6.2.3 Observer design

This section will be devoted to the parameterization of all the observer matrices and the stability analysis.

Observer parameterization

This section will focus on parameterizing all observer matrices. The full parameterization of the observer can be achieved by following the steps outlined in section 4.2.2.

Now, with the matrices N_i and S_i values obtained from section 4.2.2, matrix $\mathbb{A}_i$ can be formulated as

$$\mathbb{A}_i = \sum_{i=1}^{k} \mu_i(\varrho(t))(\mathbb{A}_{1i} - \mathbb{Y}_i\mathbb{A}_2) \tag{6.34}$$

where $\mathbb{A}_{1i} = \begin{bmatrix} N_{1i} & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{Y}_i = \begin{bmatrix} Y_{1i} & -H_i \\ -Y_{2i} & L_i \end{bmatrix}$ and $\mathbb{A}_{2_0} = \begin{bmatrix} N_2 & 0 \\ 0 & -I \end{bmatrix}$.

without loss of generality, $Y_{3i} = 0$ is taken for simplicity.

The problem of the observer design is reduced to determine matrix $\mathbb{Y}_i$ such that system (6.34) is asymptotically stable.

6.2.4 Stability analysis of the observer

This section is devoted to the stability analysis of the observer. The following theorem gives the condition for the stability in a LMI.

Theorem 5. *Under Assumptions 9 and 10, there exist a matrix $\mathbb{Y}_i$ such that system (6.34) is asymptotically stable if there exists a positive definite matrix $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} > 0$ such that the following LMI is satisfied*

$$N_2^{T\perp}[N_{1i}^T X_1 + X_1 N_{1i}]N_2^{T\perp T} < 0, \tag{6.35}$$

matrices N_{1i} and N_2 are defined in section 4.2.2. Then, the parameter matrix $\mathbb{Y}_i$ is obtained as shown below

$$\mathbb{Y}_i = X^{-1}(-\sigma\mathcal{B}^T + \sqrt{\sigma}\mathcal{L}\Gamma^{1/2})^T, \tag{6.36}$$

where

$$\sigma\mathcal{B}\mathcal{B}^T - \mathcal{Q} > 0 \tag{6.37}$$

Matrix $\mathcal{Q} = \begin{bmatrix} N_{1i}^T X_1 + X_1 N_{1i} & 0 \\ 0 & 0 \end{bmatrix}$, $\mathcal{B} = -\mathbb{A}_{2_0}^T$ *and* $\Xi = \Phi$. $\mathcal{L}$ *is a matrix with arbitrary elements such that* $||\mathcal{L}|| < 1$ *and* σ *is a positive scalar such that* $\Gamma > 0$.

Proof. Consider the following Lyapunov candidate function

$$V(\eta(t)) = \eta^T(t)X\eta(t), \tag{6.38}$$

The derivative of (6.38) along the trajectory of (6.27) gives

$$\dot{V}(\eta(t)) \leq \eta^T(t)\Big[\mathbb{A}_i^T X + X\mathbb{A}_i\Big]\eta(t), \tag{6.39}$$

or

$$\mathbb{A}_i^T X + X\mathbb{A}_i < 0, \tag{6.40}$$

replacing (6.34) in equation (6.40) we have

$$\mathbb{A}_{1i}^T X - \mathbb{A}_{2_0}^T \Phi + X\mathbb{A}_{1i} - \Phi^T \mathbb{A}_{2_0} < 0, \tag{6.41}$$

with $\Phi_i = \mathbb{Y}_i^T X$.
by incerting $\mathbb{A}_{1i}$ and X into (6.41) we have

$$\mathcal{Q}_i - \mathbb{A}_{2_0}^T \Phi_i - \Phi_i^T \mathbb{A}_{2_0} < 0, \tag{6.42}$$

Equation (6.42) can be written as

$$\mathcal{Q}_i + \mathcal{B}\Xi + (\mathcal{B}\Xi)^T < 0, \tag{6.43}$$

According to the elimination lemma (Skelton et al., 1997), there exists a matrix Ξ that satisfies (6.43) if and only if the following inequality is verified:

$$\mathcal{B}^\perp \mathcal{Q}_i \mathcal{B}^{\perp T} < 0, \tag{6.44}$$

with $\mathcal{B}^\perp = -\mathbb{A}_{2_0}^{T\perp} = \begin{bmatrix} -N_2^{T\perp} & 0 \end{bmatrix}$. Using the definition of $\mathcal{Q}_i$, (6.35) is obtained. If (6.44) is satisfied, the parameter matrix $\mathbb{Y}_i$ is obtained as in (6.36) which complete the proof of the theorem. □

Finally, the generalized adaptive observer can be obtained with the following algorithm.

1. Select a matrix $R \in \mathbb{R}^{q_0 \times n}$ such that $\text{rank}(\Sigma) = n$.
2. Compute matrices N_{1i}, N_2, F_{1i}, F_2, Q_1, T, K and P_1 defined in Section 6.2.3.
3. Solve LMI (6.35) to find matrices X and Φ_i.
4. Choose matrix $||\mathcal{L}|| < 1$ and a scalar $\sigma > 0$ such that $\Gamma > 0$.
5. Determine matrix $\mathbb{Y}_i$ using (6.36), to obtain Y_{1i}, Y_{2i}, H_i and L_i.
6. Compute different observer matrices N_i, S_i, M_i, P, Q, F_i and J_i, by using (4.25) to compute N_i, (4.31)-(4.34) to compute S_i, M_i, P and Q taking matrix $Y_{3i} = 0$. The matrix F_i are given by (4.27) and matrix J_i from (6.11).
7. Compute in real time $\Upsilon(t)$ and $\beta(t)$ using (6.15) and (6.9).

6.2.5 Numerical example

The following numerical example is chosen to illustrate the above Theorem 5 for a system with unknown input. Consider the descriptor system (6.1) with one unknown parameter $\theta(t)$ to estimate, described by:

$$
\begin{gathered}
E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & \rho(t) & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \\
G = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \Psi = \begin{bmatrix} 0 \\ \phi(t) \\ 0 \\ 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.
\end{gathered} \tag{6.45}
$$

Matrix $E^{\perp}$ such that $E^{\perp}E = 0$ is obtained as:

$$
E^{\perp} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.46}
$$

We select matrix $R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times 10^2$, such that $\text{rank}(\Sigma) = 4$.

Defining $\rho(t) \in \mathbb{R}^n$ as the variant parameters were its maximum value is $\rho = 3$ and its minimum value $\rho = 1$. By applying the above results to this system and computing N_{1i}, N_2, F_{1i}, F_2, Q_1, T, K and P_1. after resolving LMI (76), Considering $\mathcal{L} =$ $\text{ones}_{12,8} \times 0.1$ and $\sigma = 1000$ satisfying the following conditions $||\mathcal{L}|| < 1$, and $\Gamma > 0$. Matrices N_i, H_i, F_i, S_i, L_i, M_i, and J_i are computed.

$$N_1 = \begin{bmatrix} -1.060 & 0.484 & 0.522 & 1.148 \\ 0.945 & -0.809 & -0.694 & -0.935 \\ -0.054 & -0.809 & -0.744 & -0.935 \\ 1.505 & -1.294 & -1.116 & -2.084 \end{bmatrix}, \quad N_2 = \begin{bmatrix} -1.060 & 0.484 & 1.972 & 1.148 \\ 0.945 & -0.809 & -0.694 & -0.935 \\ -0.054 & -0.809 & -2.194 & -0.935 \\ 1.505 & -1.294 & 0.3331 & -2.0844 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} -5 & 5 & 16.711 & -3.088 \\ 0 & 0 & 14.793 & 5.778 \\ 5 & -5 & -32.711 & 8.278 \\ -5 & 5 & -31.920 & 3.866 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -150 & 150 & -55.78 & -75.58 \\ 0 & 0 & 14.79 & 5.77 \\ 150 & 39.78 & 39.78 & 80.77 \\ -150 & 150 & 104.42 & -68.63 \end{bmatrix},$$

$$S_1 = S2 \begin{bmatrix} 0.002 & -0.484 & -0.239 & -0.486 \\ 0.002 & -0.484 & -0.239 & -0.486 \\ 0.002 & -0.484 & -0.239 & -0.486 \\ 0.002 & -0.484 & -0.239 & -0.486 \end{bmatrix} \times 10^{-3}, \quad M_1 = M_2 \begin{bmatrix} 0 & 0 & 0.012 & 0.012 \\ 0 & 0 & 0.012 & 0.012 \\ 0 & 0 & 0.012 & 0.012 \\ 0 & 0 & 0.012 & 0.012 \end{bmatrix}, \quad J = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

In order to simulate the considered linear descriptor system and the proposed adaptive observer, the input $\phi(t)$ is chosen to be a square impulses as shown in Figure 6.1. For clearness reasons, only the first 100 seconds are presented. This signal is rich and satisfies the persistent excitation condition.

The true parameter to estimate $\theta(t)$ switches between 1 and 2. The initial values for the state vector, its estimate and parameter estimate are: $x(0) = \begin{bmatrix} -1 & -1 & -1 & 2 \end{bmatrix}^T$; $\hat{x}(0) \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$; $\hat{\theta}(0) = 0$. The adaptive observer parameters are: $\Lambda = ones_{12\times 8}$ and $\Gamma = 100$.

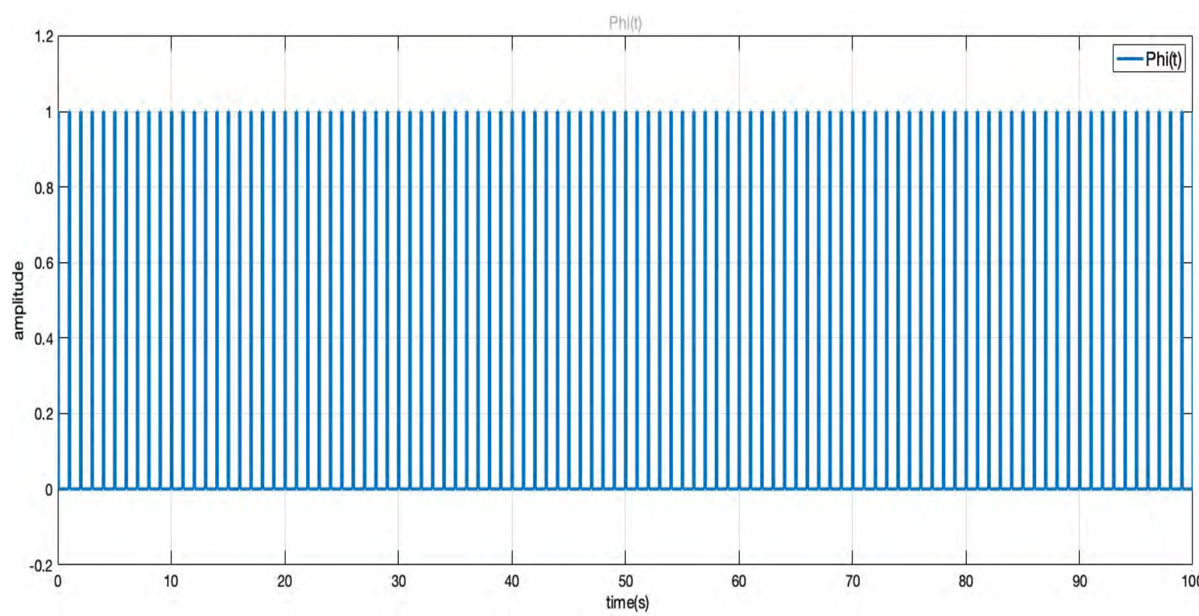

Fig. 6.1. Input signal $\phi(t)$ for LPV systems.

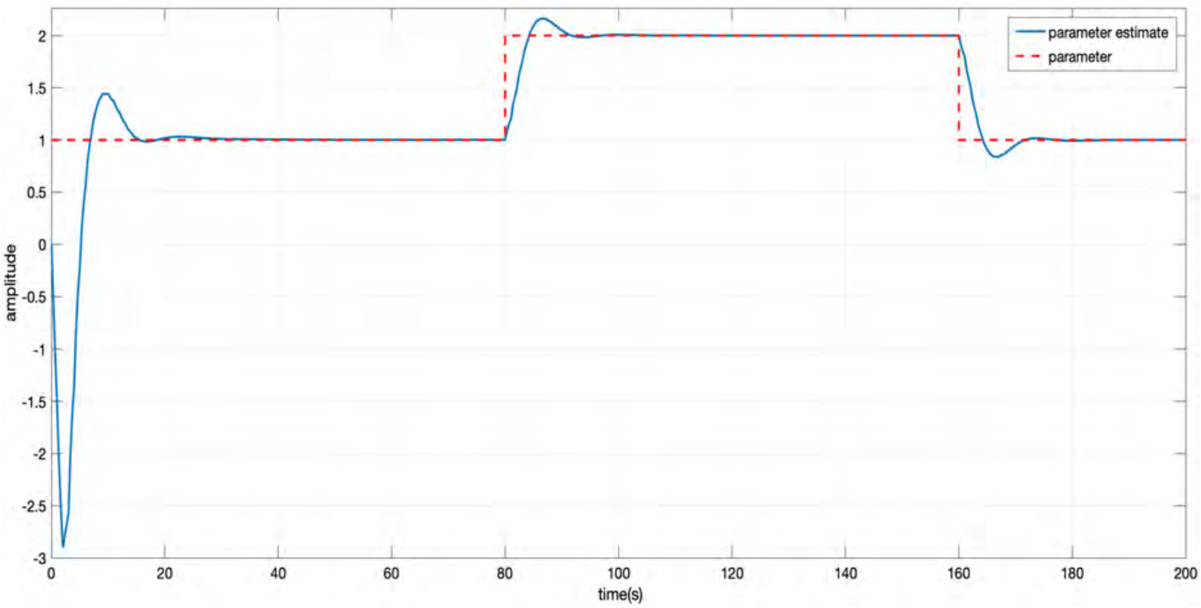

Fig. 6.2. Convergence behavior of parameter estimate $\theta(t)$ for LPV systems

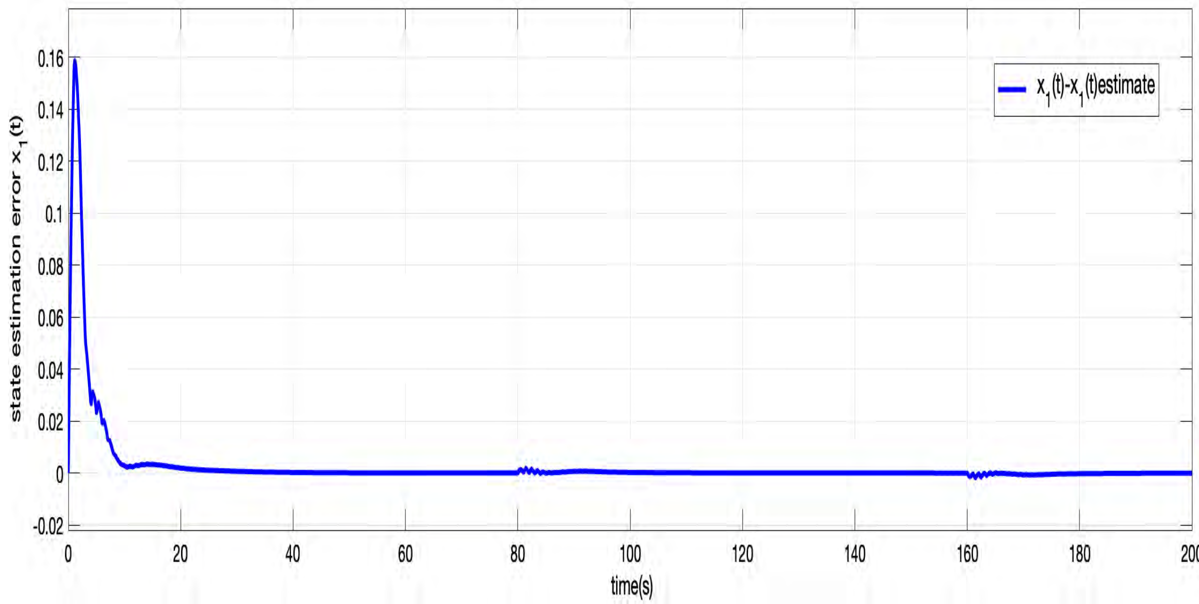

Fig. 6.3. Convergence behavior of state estimation error $x_1(t)$ for LPV systems

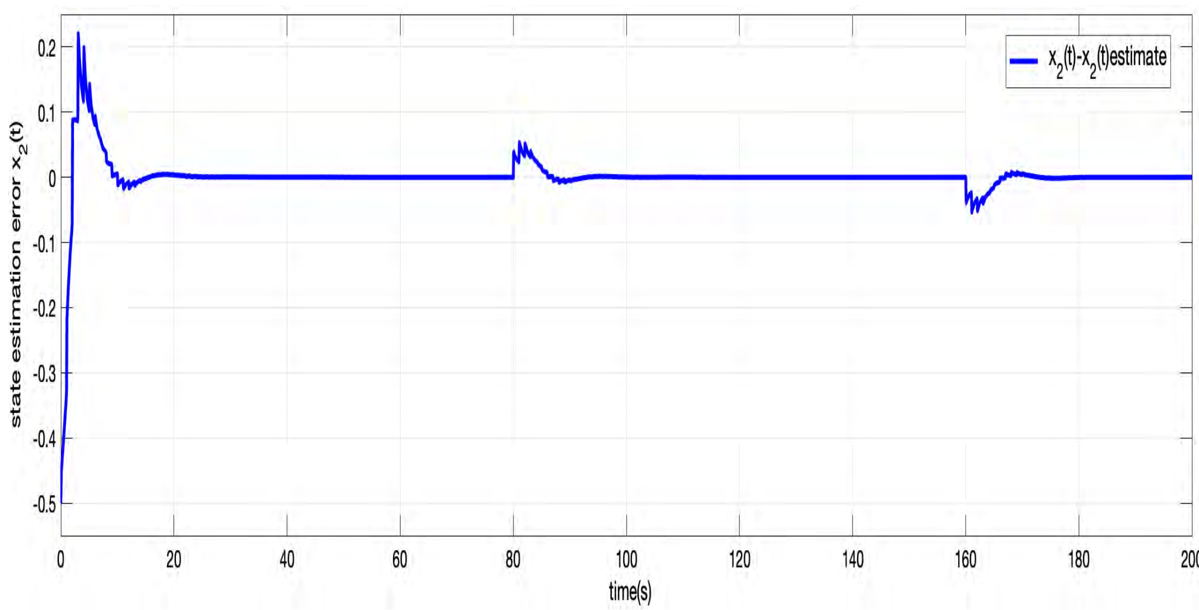

Fig. 6.4. Convergence behavior of state estimation error $x_2(t)$ for LPV systems

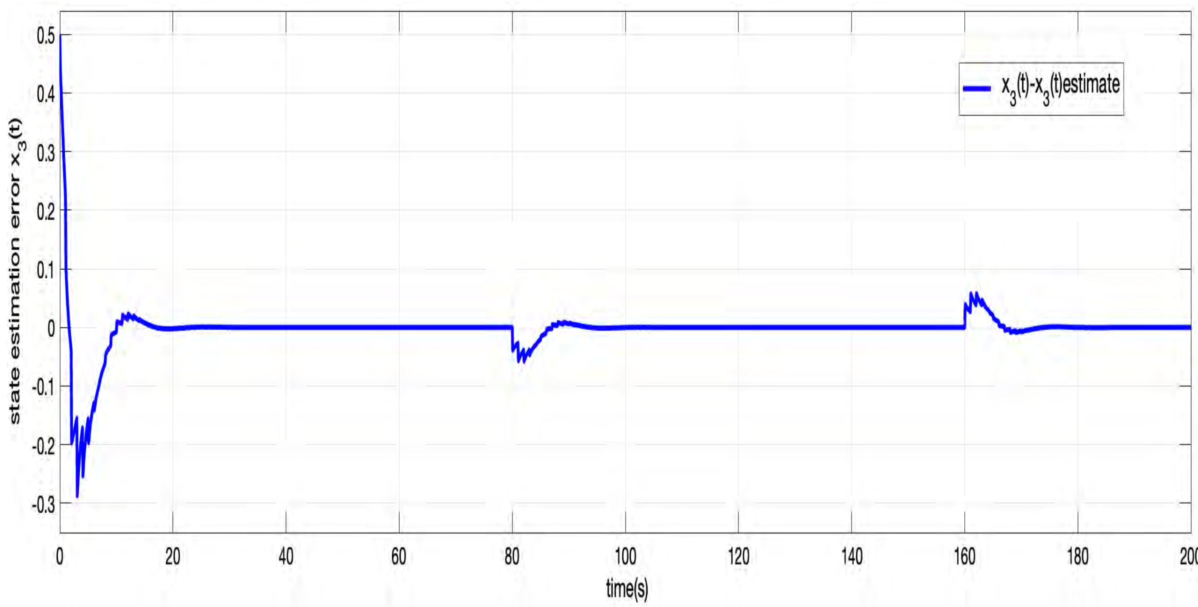

Fig. 6.5. Convergence behavior of state estimation error $x_3(t)$ for LPV systems

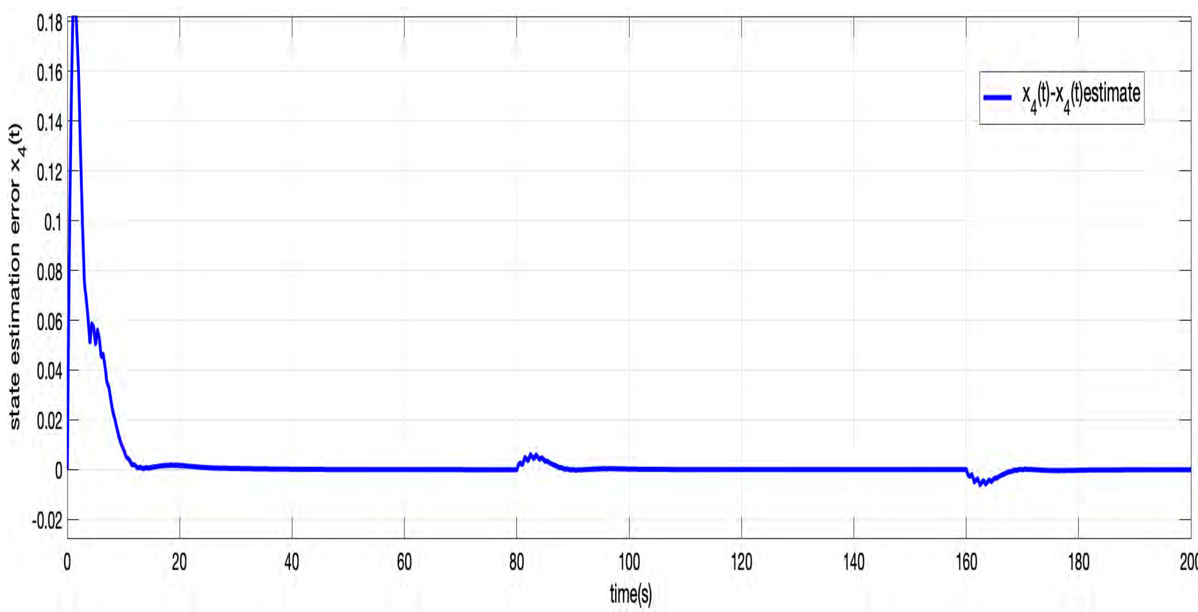

Fig. 6.6. Convergence behavior of state estimation error $x_4(t)$ for LPV systems

Figure 6.2 presents the parameter estimation in relation to its original value. One can see that the estimate converges fastly to its real value. he rate has been adjusted with a good choice of parameters Λ and Γ. Figures 6.3-6.6 shows the states estimation errors evolutions $e_x(t)$ for all the state vector. The observer permits a successful convergence to the states of the system. It can be shown that the state estimation error converges fastly to 0 after each change in $\theta(t)$ which demonstrate the effectiveness of the proposed approach.

6.2.6 Conclusions

In this chapter we propose a novel generalized adaptive observer design for joint estimation of states and parameters for algebro-differential linear and algebro-differential linear parameter varying systems. The designs are given in term of LMIs. these approaches represents a generalization of PIO and PO methods, enabling the concurrent estimation of some system parameters and state variables, given that specific rank relations and a persistent condition are fulfilled. In order to illustrate the efficiency of the proposed approach, A numerical example has been included to illustrate the performance of the proposed adaptive observer design.

Chapter 7

Conclusions and perspectives

This thesis has contributed to algebro-differential parameter-varying systems in different categories as state estimation, parameter estimation and fault diagnosis. Based on these categories, a bibliographical review is presented to guide the direction of this work.

Chapter 3 focuses on the GDO synthesis for algebro-differential parameter-varying systems systems. It is introduced the GDO structure considering the case for state estimation. This type of structure takes into account a wider range of systems by combining it with a more general observer structure, allowing for a more comprehensive methodology. In the case of variable parameters, conditions for the existence of the GDO are provided, and its stability has been demonstrated to tackle this issue. The stability conditions are formulated in terms of LMIs, utilizing the bounded real lemma and other mathematical complements. Additionally, the designs include detailed parameterizations of the algebraic constraints.

In the same way, Chapter 4 concerns simultaneous state variables and unknown inputs estimations for algebro-differential parameter-varying systems through the design of a generalized observer. Additionally, the design of a generalized learning observer is presented, which allows reducing the convergence time for fault estimation applied to the same class of systems.

The state and parameter estimation are addressed in Chapter 5 for algebro-differential linear systems, using the GDO structure, allowing for the simultaneous estimation of system parameters and state variables once some rank relations and persistent conditions are satisfied. Additionally, the case for algebro-differential LPV systems is presented for a broader operational range.

Different applications are presented in each observer design to demonstrate the performance of each one. It can be noted that the performance of the GDO achieves a better performance index in comparison to their particular structures (PO, PIO).

Along with this research work, many open problems were detected, giving opportunities for more contributions to the topics encompassed in this thesis. Some of the open problems are presented below :

- The algebro-differential nonlinear parameter-varying systems are a very interesting class of systems studied in this research. The solutions provided in the designs could be conservative if there is a considerable amount of noise in the system, in addition to considering an estimation of time-varying unknown inputs. It can take into account those scenarios, improving upon the observer designs previously demonstrated.

- The design of adaptive observers in Chapter 5 requires a persistent excitation condition, which must be satisfied by specific signals to estimate the unknown parameters. A case where this condition is not taken into account for parameter estimation applied to the same type of systems can be considered.

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Appendix

Appendix A

Performance index IAE, ISE, and ITAE

A performance index is a quantitative measure of a system's behavior and is chosen in a way that highlights the system's important specifications (Dorf et al. (2005)).

A system is considered optimal when its parameters are adjusted in such a way that the index reaches an extreme value, typically a minimum value. For a performance index to be useful, it must always be a positive number or zero, such that the best behavior is defined as minimizing this index.

Various performance indices (Dorf et al. (2005)) are employed in the literature, and the selection of such indices depends on the type of behavior to be analyzed and the characteristics of the system.

The Integral Squared Error (ISE) is expressed as:

$$ISE = \int_0^t e^2(t)dt \tag{A.0.1}$$

This index penalizes large errors and discriminates between excessively overdamped and underdamped responses. The minimum value of the integral occurs for a critical damping value.

Another index used to reduce the contribution of the initial error and highlight errors occurring after the response (Graham and Lathrop (1953)) is the value of the integral of time multiplied by the absolute error (ITAE), defined as follows:

$$ITAE = \int_0^t t|e(t)|dt \tag{A.0.2}$$

Another particularly useful index for simulation studies is the value of the integral of the absolute error (IAE). It is a more sensitive index than ISE, and therefore, IAE tends to give longer settling times and higher overshoots. IAE is defined as:

$$IAE = \int_0^t |e(t)|dt \tag{A.0.3}$$

There are other performance indices besides those described above. However, for the purposes of this work, the aforementioned ones are employed.

Appendix B

Generalized dynamic adaptive observer design for parameter estimation

B.1 Preliminaries

Let us consider the following linear time invariant descriptor system:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + \psi(t)\theta(t), \\ y(t) &= Cx(t), \end{aligned} \tag{B.1.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the known input, $y(t) \in \mathbb{R}^p$ is the measurement output vector and $\theta(t) \in \mathbb{R}^l$ is the unknown parameter vector assumed to be constant. Matrix $E \in \mathbb{R}^{n\times n}$ could be singular. Matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are known constant matrices. $\psi \in \mathbb{R}^{n\times l}$ is a matrix of known signals, and it is assumed to be piecewise differentiable. Both $\psi(t)$ and its derivative $\dot{\psi}(t)$ are uniformly bounded in time. Let $rank(E) = r < n$ and $E^{\perp} \in \mathbb{R}^{s\times n}$ be a full row matrix such that $E^{\perp}E = 0$, in this case $s = n - r$. Without loss of generality: $rank(C) = p$.

The problem addressed in this paper involves the simultaneous estimation of $x(t)$ and $\theta(t)$ using measured signals $u(t)$, $y(t)$, and $\psi(t)$.

Considering the aforementioned, system (B.1.1) is equivalent to:

$$\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} x(t) + \begin{bmatrix} E^{\perp}\psi(t) \\ 0 \end{bmatrix} \theta(t) \tag{B.1.2}$$

Our aim is to design an adaptive observer of the form:

$$\dot{\zeta}(t) = N\zeta(t) + Hv(t) + F\left(\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} - \begin{bmatrix} E^{\perp}\psi(t) \\ 0 \end{bmatrix}\hat{\theta}(t)\right) + Ju(t) + T\psi\hat{\theta}(t) + \varUpsilon_1\dot{\hat{\theta}}(t), \tag{B.1.3}$$

$$\dot{v}(t) = S\zeta(t) + Lv(t) + M\left(\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} - \begin{bmatrix} E^{\perp}\psi(t) \\ 0 \end{bmatrix}\hat{\theta}(t)\right), \tag{B.1.4}$$

$$\hat{x}(t) = P\zeta(t) + Q\left(\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} - \begin{bmatrix} E^{\perp}\psi(t) \\ 0 \end{bmatrix}\hat{\theta}(t)\right), \tag{B.1.5}$$

$$\dot{\hat{\theta}}(t) = \Gamma\beta^T(t)C^T\varLambda(y(t) - C\hat{x}(t)), \tag{B.1.6}$$

where $\hat{x}(t)$ and $\hat{\theta}(t)$ are the estimates of $x(t)$ and $\theta(t)$, respectively. The matrices N, H, F, S, L, M, J, P, Q, T, Γ and $\beta(t)$ are unknown matrices of appropriate dimensions, which must be determined such that $\hat{x}(t)$ converges asymptotically to $x(t)$ and $\hat{\theta}(t)$ converges to $\theta(t)$, respectively. Γ and $\varLambda$ are symmetric positive matrices which are used to adjust the evolution rate of $\hat{\theta}(t)$.

B.2 Problem statement

Let us consider the two estimation errors

$$\begin{aligned} e_x(t) &= x(t) - \hat{x}(t). \\ e_\theta(t) &= \theta(t) - \hat{\theta}(t). \end{aligned} \tag{B.2.1}$$

Now, we make the following two assumptions which are needed for the rest of the paper

Assumption 11. *It is assumed that the system described by equation* (B.1.1) *is regular (Definition 1), Impulse observable (Definition 2) and Reachable observable (Definition 4).*

Assumption 12. *let $\beta(t) \in \mathbb{R}^{n\times l}$ be a matrix of signals generated by the stable ODE system:*

$$\dot{\beta}(t) = \mathbb{A}\beta(t) + \mathcal{X}(t). \tag{B.2.2}$$

Assuming that $\mathcal{X}$ is persistently exciting, there exist two positive constants, δ, t_1 and some bounded symmetric positive definite matrix $\varLambda \in \mathbb{R}^{q\times q}$ such that for all t the following inequality holds:

$$\int_t^{t-t_1} \beta^T(\tau)C^T\varLambda C\beta(\tau)d\tau \geq \delta I. \tag{B.2.3}$$

Remark 2. *Condition (B.2.3) is typically required for parameter identification.*
The following theorem gives the conditions for the existence of the observer:

Theorem 6. *Let $\Gamma \in \mathbb{R}^{l\times l}$ be any symmetric positive definite matrix. Under Assumption* 11, *and for a constant $\theta(t)$, the adaptive observer (B.1.3) - (B.1.6), is a global exponential adaptive observer for the descriptor system (B.1.1) if and only if there exists a matrix T such that:*

$$
\begin{aligned}
J &= TB, && \text{(B.2.4)}\\
NTE + F\begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} &= TA, && \text{(B.2.5)}\\
STE + M\begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} &= 0, && \text{(B.2.6)}\\
PTE + Q\begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} &= I_n, && \text{(B.2.7)}
\end{aligned}
$$

and $\mathbb{A} = \begin{bmatrix} N & -H \\ -S & L \end{bmatrix}$ *is a stable matrix.*

The matrix of signals $\Upsilon(t) = \begin{bmatrix} \Upsilon_1(t) \\ 0 \end{bmatrix}$ *is obtained by linearly filtering $\psi(t)$ through:*

$$\dot{\Upsilon}(t) = \mathbb{A}\Upsilon(t) + \mathbb{D}\psi(t), \tag{B.2.8}$$

where $\mathbb{D} = \begin{bmatrix} T - F\begin{bmatrix} E^{\perp} \\ 0 \end{bmatrix} \\ M\begin{bmatrix} E^{\perp} \\ 0 \end{bmatrix} \end{bmatrix}$.

The matrix $\beta(t)$ is given from equation (B.2.2) such that :

$$\mathcal{X}(t) = [\mathbb{A}Q\begin{bmatrix} E^{\perp} \\ 0 \end{bmatrix} + \mathbb{D}]\psi(t) - Q\begin{bmatrix} E^{\perp} \\ 0 \end{bmatrix}\dot{\psi}(t), \tag{B.2.9}$$

or by:

$$\beta(t) = \mathbb{P}\Upsilon(t) - Q\begin{bmatrix} E^{\perp}\psi(t) \\ 0 \end{bmatrix}, \tag{B.2.10}$$

with $\mathbb{P} = \begin{bmatrix} P & 0 \end{bmatrix}$.

Proof. Let us define the transformation error $\epsilon(t)$ as:

$$\epsilon(t) = TEx(t) - \zeta(t), \tag{B.2.11}$$

Its derivative is given by

$$
\begin{aligned}
\dot{\epsilon}(t) = N\epsilon(t) - Hv(t) - \left(NTE - TA + F\begin{bmatrix} E^{\perp}A \\ C \end{bmatrix}\right)x(t) + \\
(TB - J)u(t) + \left(T - F\begin{bmatrix} E^{\perp} \\ 0 \end{bmatrix}\right)\psi(t)e_{\theta}(t) - \Upsilon_1\dot{\hat{\theta}}(t),
\end{aligned}
\tag{B.2.12}
$$

Using (B.2.4) and (B.2.5), we can obtain:

$$\dot{\epsilon}(t) = N\epsilon(t) - Hv(t) + \left(T - F\begin{bmatrix} E^{\perp} \\ 0 \end{bmatrix}\right)\psi(t)e_\theta(t) + \Upsilon_1 \dot{e}_\theta(t), \tag{B.2.13}$$

Now, by utilizing (B.2.11), equation (B.1.4) can be expressed as:

$$\dot{v}(t) = -S\epsilon(t) + Lv(t) + (STE + M\begin{bmatrix} E^{\perp}A \\ C \end{bmatrix})x(t) + M\begin{bmatrix} E^{\perp} \\ 0 \end{bmatrix}\psi(t)e_\theta(t), \tag{B.2.14}$$

Using (B.2.6) we have

$$\dot{v}(t) = -S\epsilon(t) + Lv(t) + M\begin{bmatrix} E^{\perp} \\ 0 \end{bmatrix}\psi(t)e_\theta(t). \tag{B.2.15}$$

Considering equations (B.2.12) and (B.2.14), the following observer error dynamics equation is obtained

$$\dot{\varphi}(t) \;\; = \;\; \mathbb{A}\varphi(t) + \mathbb{D}e_\theta(t) + \Upsilon(t)\dot{e}_\theta(t), \tag{B.2.16}$$

where $\varphi(t) = \begin{bmatrix} \epsilon(t) \\ v(t) \end{bmatrix}$.

The key step of the proof is to define the following linear combination of $\varphi(t)$ and $e_\theta(t)$:

$$\eta(t) = \varphi(t) - \Upsilon(t)e_\theta(t), \tag{B.2.17}$$

then we have:

$$\dot{\eta}(t) = \dot{\varphi}(t) - \dot{\Upsilon}(t)e_\theta(t) - \Upsilon(t)\dot{e}_\theta(t). \tag{B.2.18}$$

By substituting equation (B.2.16) into equation (B.2.18), we obtain:

$$\dot{\eta}(t) = \mathbb{A}\eta(t) + \left(\mathbb{D}\psi(t) + \mathbb{A}\Upsilon(t) - \dot{\Upsilon}(t)\right)e_\theta(t). \tag{B.2.19}$$

From linear equation (B.2.8), equation (B.2.19) is reduced to its homogeneous part:

$$\dot{\eta}(t) = \mathbb{A}\eta(t). \tag{B.2.20}$$

Matrix $\mathbb{A}$ must be a stability matrix to ensure convergence of $\eta(t)$ to 0.
The first result of the proof is that, since $\Upsilon(t)$ is generated by (B.2.8) and under above conditions for matrices $\mathbb{A}$ and $\mathbb{D}$, system (B.2.19) generating $\eta(t)$ is globally exponentially stable, so $\eta(t)$ converges to 0 with global and exponential convergence.

Now and since the parameter vector $\theta(t)$ is constant, we have:

$$\dot{\theta}(t) = 0, \tag{B.2.21}$$

and

$$
\begin{aligned}
\dot{e}_\theta(t) &= \dot{\theta}(t) - \dot{\hat{\theta}}(t), \\
&= -\Gamma\beta^T(t)C^T\Lambda(y(t) - \hat{y}(t)), \\
&= -\Gamma\beta^T(t)C^T\Lambda Ce_x(t).
\end{aligned} \tag{B.2.22}
$$

On the other hand, by taking (B.2.7), we can easily prove that

$$
e_x(t) = \mathbb{P}\varphi(t) - Q\begin{bmatrix} E^\perp \\ 0 \end{bmatrix}\psi(t)e_\theta(t), \tag{B.2.23}
$$

Then, by replacing $\varphi(t)$ from (B.2.17), and using (B.2.23) in (B.2.22), we obtain:

$$
\dot{e}_\theta = -\Gamma\beta^T(t)C^T\Lambda C\left[\mathbb{P}\eta(t) + [\mathbb{P}\Upsilon(t) - Q\begin{bmatrix} E^\perp\psi(t) \\ 0 \end{bmatrix}]e_\theta(t)\right], \tag{B.2.24}
$$

by choosing $\beta(t)$ as in (B.2.10), equation (B.2.24) becomes:

$$
\dot{e}_\theta = -\Gamma\beta^T(t)C^T\Lambda C\left[\mathbb{P}\eta(t) + \beta(t)e_\theta(t)\right], \tag{B.2.25}
$$

Looking to the homogeneous part of system (B.2.25) leads to:

$$
\dot{e}_\theta = -\Gamma\beta^T(t)C^T\Lambda C\beta(t)e_\theta(t). \tag{B.2.26}
$$

As $\psi(t)$ and $\dot{\psi}(t)$ are bounded, $\beta(t)$ generated from the exponentially stable system (B.2.2) is also bounded. From the persistent excitation condition (B.2.3), and since Γ and Λ are positive, the global exponential stability of system (B.2.26) can be easily proved. From the exponential convergence of $\eta(t)$ and of system (B.2.26), we obtain the global and exponential convergence to 0 of $e_\theta(t)$ generated from system (B.2.25) (see Zhang (2002), Alma and Darouach (2014) and Alma et al. (2018) for more details).

Now, from $\eta(t) \to 0$, $e_\theta(t) \to 0$ and the fact that $\Upsilon(t)$ is bounded, we conclude that $\varphi(t) = \eta(t) + \Upsilon(t)e_\theta(t)$ converges to 0, with global and exponential convergence.

Finally, from the convergence of $e_\theta(t)$ and $\varphi(t)$ to 0, and from (B.2.23), the global and exponential convergence of $e_x(t) \to 0$ is guaranteed.

□

B.3 Observer design

This section will be devoted to the parameterization of all the observer matrices and the stability analysis.

Observer parameterization

This section will be devoted to the parameterization of all matrices observers. The subsequent derivation of the observer parametrization follows a similar approach to that presented in the section 4.2.2.

Now, by using the values of matrices N and S provided by the previous parametrization, the matrix $\mathbb{A}$ can be expressed as

$$\mathbb{A} \;=\; (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2) \tag{B.3.1}$$

where $\mathbb{A}_1 = \begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{Y} = \begin{bmatrix} Y_1 & -H \\ -Y_2 & L \end{bmatrix}$ and $\mathbb{A}_2 = \begin{bmatrix} N_2 & 0 \\ 0 & -I \end{bmatrix}$.

without loss of generality, $Y_3 = 0$ is taken for simplicity.

The problem of the observer design is reduced to determine matrix $\mathbb{Y}$ such that system (B.3.1) is asymptotically stable.

B.4 Stability analysis of the observer

This section is devoted to the stability analysis of the observer. The following theorem gives the condition for the stability in a LMI.

Theorem 7. *Under Assumptions 11, there exists a matrix $\mathbb{Y}$ such that system* (B.2.20) *is asymptotically stable if there exists a positive definite matrix $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} > 0$ such that the following LMI is satisfied*

$$N_2^{T\perp}[N_1^T X_1 + X_1 N_1] N_2^{T\perp T} < 0 \tag{B.4.1}$$

matrices N_1 and N_2 are defined in section 4.2.2. Then, the parameter matrix $\mathbb{Y}$ is obtained as shown below

$$\mathbb{Y} = X^{-1}(-\sigma\mathcal{B}^T + \sqrt{\sigma}\mathcal{L}\Delta^{1/2})^T, \tag{B.4.2}$$

where

$$\sigma\mathcal{B}\mathcal{B}^T - \mathcal{Q} > 0 \tag{B.4.3}$$

Matrix $\mathcal{Q} = \begin{bmatrix} N_1^T X_1 + X_1 N_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} -N_2^T & 0 \\ 0 & I \end{bmatrix}$ and $\Xi = \Phi$. $\mathcal{L}$ is a matrix with arbitrary elements such that $||\mathcal{L}|| < 1$ and σ is a positive scalar such that $\Delta > 0$.

Proof. Consider the following Lyapunov candidate function

$$V(\eta(t)) = \eta^T(t) X \eta(t), \tag{B.4.4}$$

The derivative of (B.4.4) along the trajectory of (B.2.20) gives

$$\dot{V}(\eta(t)) \leq \eta^T(t)\Big[\mathbb{A}^T X + X\mathbb{A}\Big]\eta(t), \tag{B.4.5}$$

or

$$\mathbb{A}^T X + X\mathbb{A} < 0, \tag{B.4.6}$$

replacing (B.3.1) in equation (B.4.6) we have

$$\mathbb{A}_1^T X - \mathbb{A}_2^T \Phi + X\mathbb{A}_1 - \Phi^T \mathbb{A}_2 < 0, \tag{B.4.7}$$

with $\Phi = \mathbb{Y}^T X$.
by inserting $\mathbb{A}_1$ and X into (B.4.7) we have

$$\mathcal{Q} - \mathbb{A}_2^T \Phi - \Phi^T \mathbb{A}_2 < 0, \tag{B.4.8}$$

Equation (B.4.8) can be written as

$$\mathcal{Q} + \mathcal{B}\Xi + (\mathcal{B}\Xi)^T < 0. \tag{B.4.9}$$

According to the elimination lemma (Skelton et al., 1997), there exists a matrix Ξ that satisfies (B.4.9) if and only if the following inequality is verified:

$$\mathcal{B}^{\perp} \mathcal{Q} \mathcal{B}^{\perp T} < 0, \tag{B.4.10}$$

with $\mathcal{B}^{\perp} = -\mathbb{A}_2^{T\perp} = \begin{bmatrix} N_2^{T\perp} & 0 \end{bmatrix}$. Using the definition of $\mathcal{Q}$, (B.4.1) is obtained. If (B.4.10) is satisfied, the parameter matrix $\mathbb{Y}$ is obtained as in (B.4.2) which complete the proof of the theorem.

□

Finally, the generalized adaptive observer can be obtained with the following algorithm.

1. Select a matrix $R \in \mathbb{R}^{q_0 \times n}$ such that $\text{rank}(\Sigma) = n$.

2. Compute matrices N_1, N_2, F_1, F_2, Q_1, T, K and P_1 defined in Section B.3.

3. Solve LMI (B.4.1) to find matrices X and Φ.

4. Choose matrix $||\mathcal{L}|| < 1$ and a scalar $\sigma > 0$ such that $\Gamma > 0$.

5. Determine matrix $\mathbb{Y}$ using (B.4.2), to obtain Y_1, Y_2, H and L.

6. Compute different observer matrices N, S, M, P, Q, F and J, by using (4.25) to compute N, (4.31)-(4.34) to compute S, M, P and Q taking matrix $Y_3 = 0$. The matrix F are given by (4.27) and matrix J from (B.2.4).

7. Compute in real time $\Upsilon(t)$ and $\beta(t)$ using (B.2.8) and (B.2.2).

B.5 Numerical example

The following numerical example is chosen to illustrate the proposed design. Les us consider the descriptor system (B.1.1) with one unknown parameter $\theta(t)$ to estimate, described by:

$$\begin{gathered}E = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}, A = \begin{bmatrix}0 & 0 & 1 & 0\\1 & 0 & 0 & 0\\-1 & 0 & 0 & 1\\0 & 1 & 1 & 1\end{bmatrix}, B = \begin{bmatrix}0\\0\\0\\-1\end{bmatrix},\\ G = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \psi(t) = \begin{bmatrix}0\\\phi(t)\\0\\0\end{bmatrix} \text{ and } C = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{bmatrix}.\end{gathered} \tag{B.5.1}$$

Matrix $E^{\perp}$ such that $E^{\perp}E = 0$ is obtained as:

$$E^{\perp} = \begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \tag{B.5.2}$$

We select matrix $R = \begin{bmatrix}1 & 0 & 1 & 1\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \times 10^2$, such that $rank(\Sigma) = rank\left(\begin{bmatrix}R\\E^{\perp}A\\C\end{bmatrix}\right) = 4$.

By applying the above results to this system we can compute matrices N_1, N_2, F_1, F_2, Q_1, T, K and P_1. After resolving LMI (B.4.1), considering $\mathcal{L} = ones_{12,8} \times 0.1$ and $\sigma = 1000$ satisfying the following conditions $||\mathcal{L}|| < 1$, and $\Delta > 0$. Matrices N, H, F, S, L, M, and J are computed.

$$\begin{aligned}N &= \begin{bmatrix}-0.5587 & 0.0588 & 1.0585 & 0.5582\\0.5151 & -0.0531 & -0.5199 & -0.5246\\-0.4715 & 0.0475 & -0.0187 & 0.4911\\-0.0436 & 0.0057 & 0.5386 & 0.0335\end{bmatrix},\\
F &= \begin{bmatrix}-50 & 50 & -47.0486 & -27.9492\\0 & 0 & 24.4800 & 2.4176\\50 & -50 & -1.9114 & 23.1154\\-50 & 50 & -22.5686 & -25.5329\end{bmatrix},\\
S &= \begin{bmatrix}-0.0021 & -0.1345 & -0.0645 & -0.1317\\-0.0021 & -0.1345 & -0.0645 & -0.1317\\-0.0021 & -0.1345 & -0.0645 & -0.1317\\-0.0021 & -0.1345 & -0.0645 & -0.1317\end{bmatrix} \times 10^{-3},\\
M &= \begin{bmatrix}0 & 0 & 0.0035 & 0.0034\\0 & 0 & 0.0035 & 0.0034\\0 & 0 & 0.0035 & 0.0034\\0 & 0 & 0.0035 & 0.0034\end{bmatrix}, \; J = \begin{bmatrix}0\\0\\0\\0\end{bmatrix},\end{aligned}$$

In order to simulate the considered linear descriptor system and the proposed adaptive observer, the input $\phi(t)$ is chosen to be a square impulses as shown in Figure B.5.1. For clearness reasons, only the first 100 seconds are presented. This signal is rich and satisfies the persistent excitation condition.

The true parameter to estimate $\theta(t)$ switches between 0.5 and 1.5. The initial values for the state vector are: $x(0) = \begin{bmatrix} -1 & -1 & -1 & 2 \end{bmatrix}^T$; $\hat{x}(0) \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$; $\hat{\theta}(0) = 0$. The adaptive observer parameters are: $\Lambda = ones_{12\times 8}$ and $\Gamma = 100$.

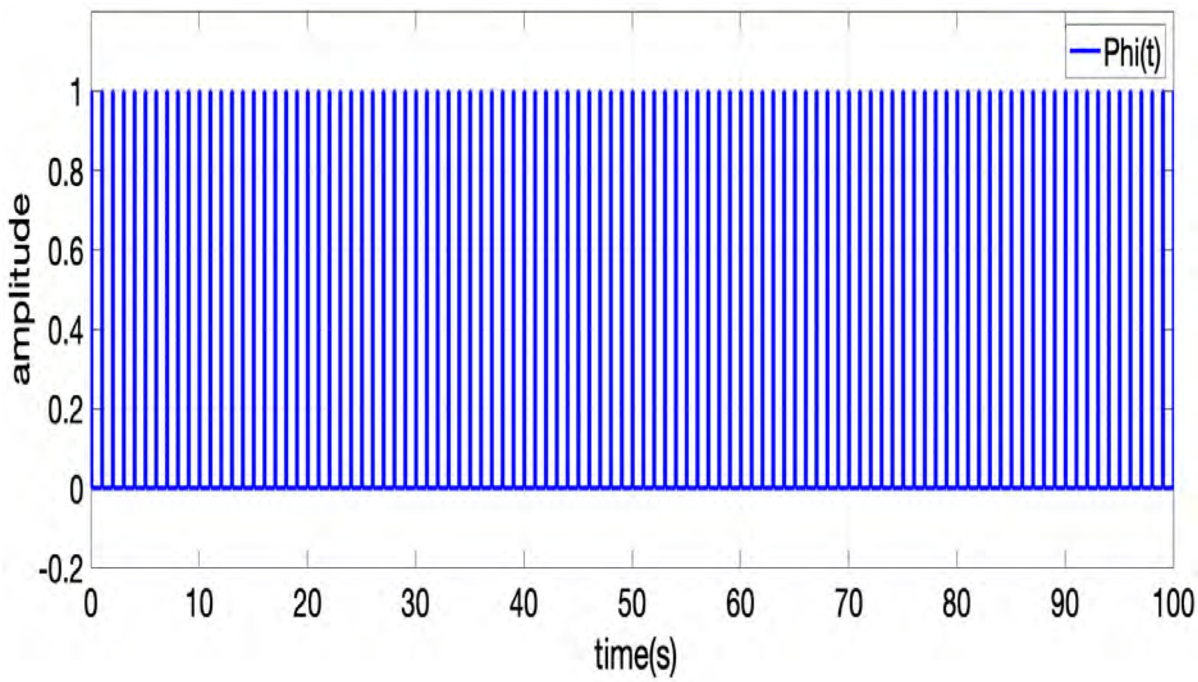

Fig. B.5.1. Simulated input $\phi(t)$.

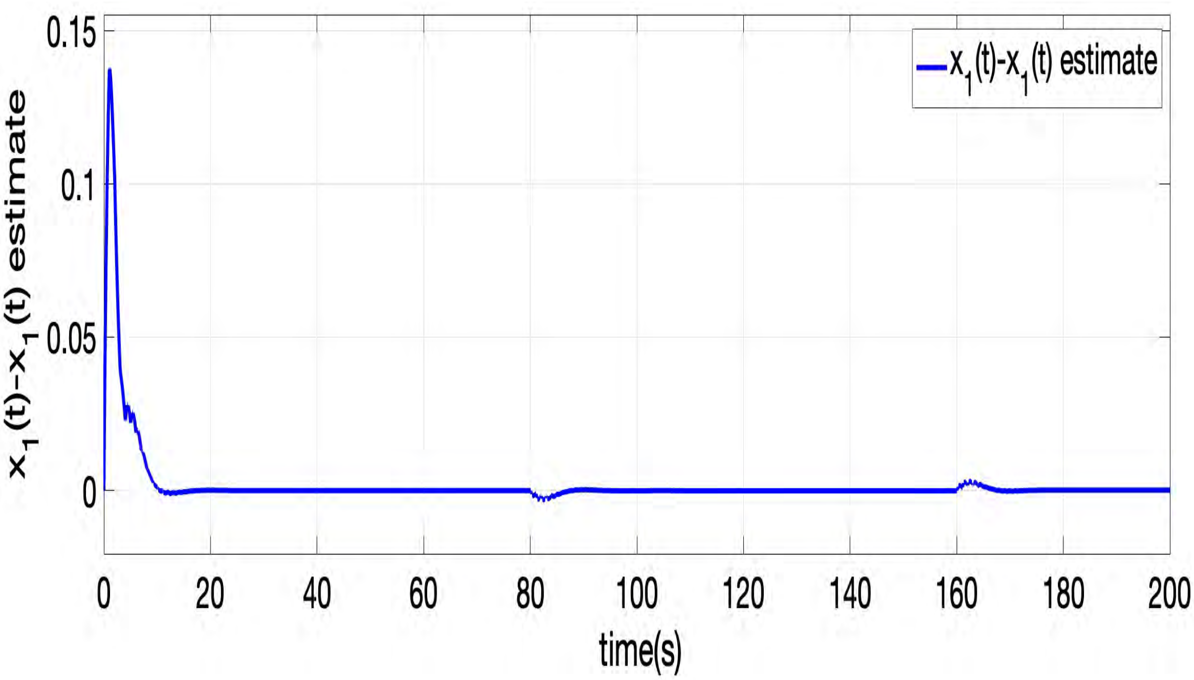

Fig. B.5.2. State estimation error $x_1(t)$ convergence.

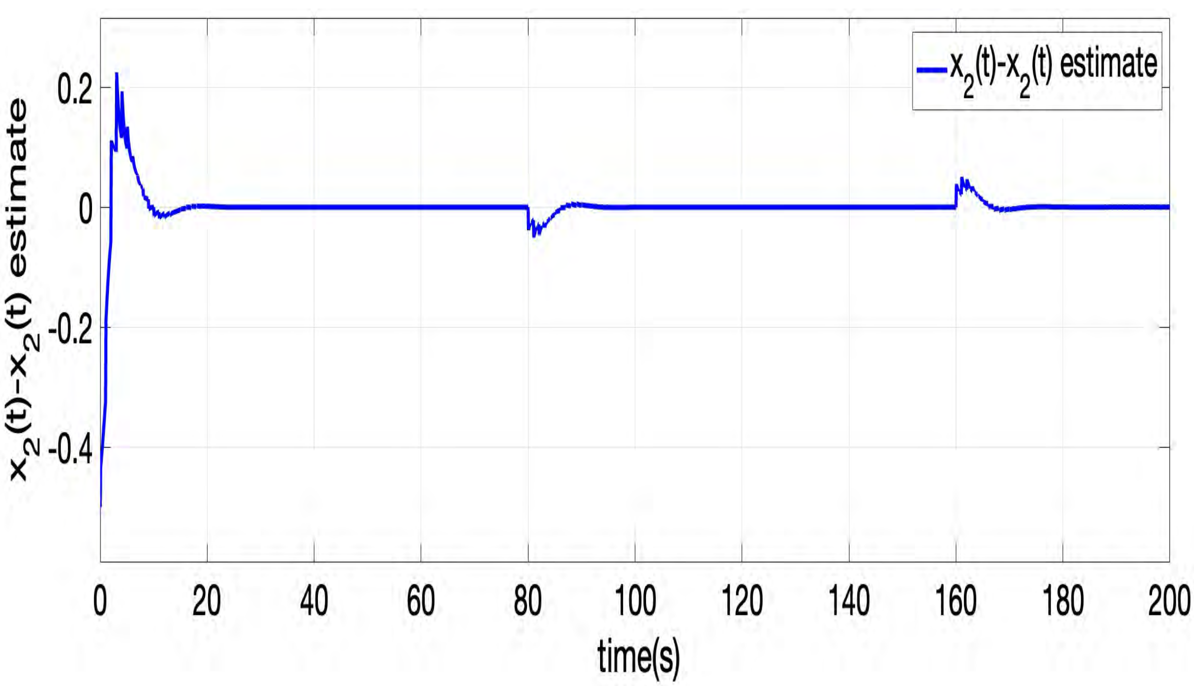

Fig. B.5.3. State estimation error $x_2(t)$ convergence.

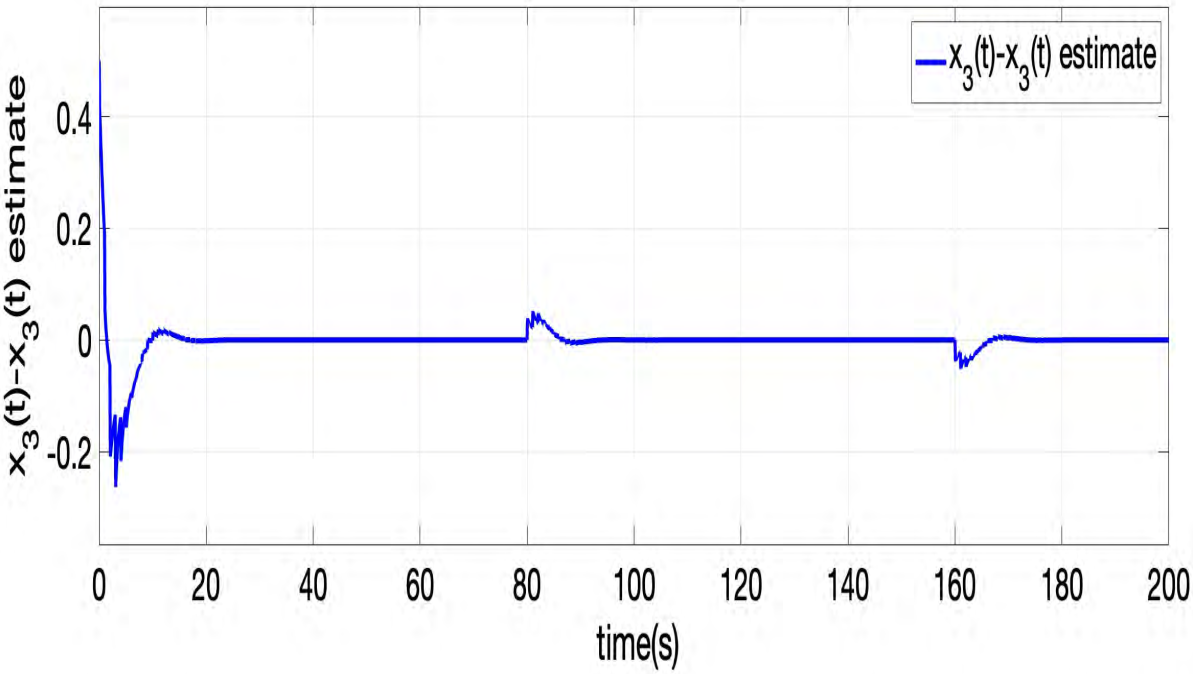

Fig. B.5.4. State estimation error $x_3(t)$ convergence.

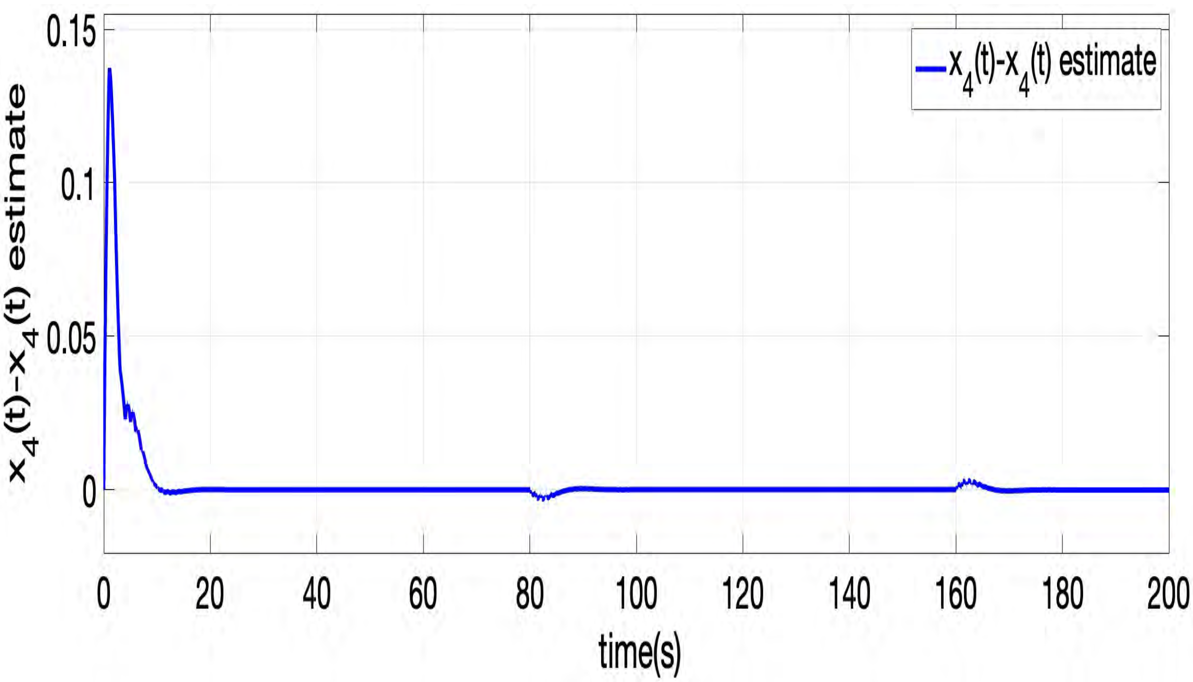

Fig. B.5.5. State estimation error $x_4(t)$ convergence.

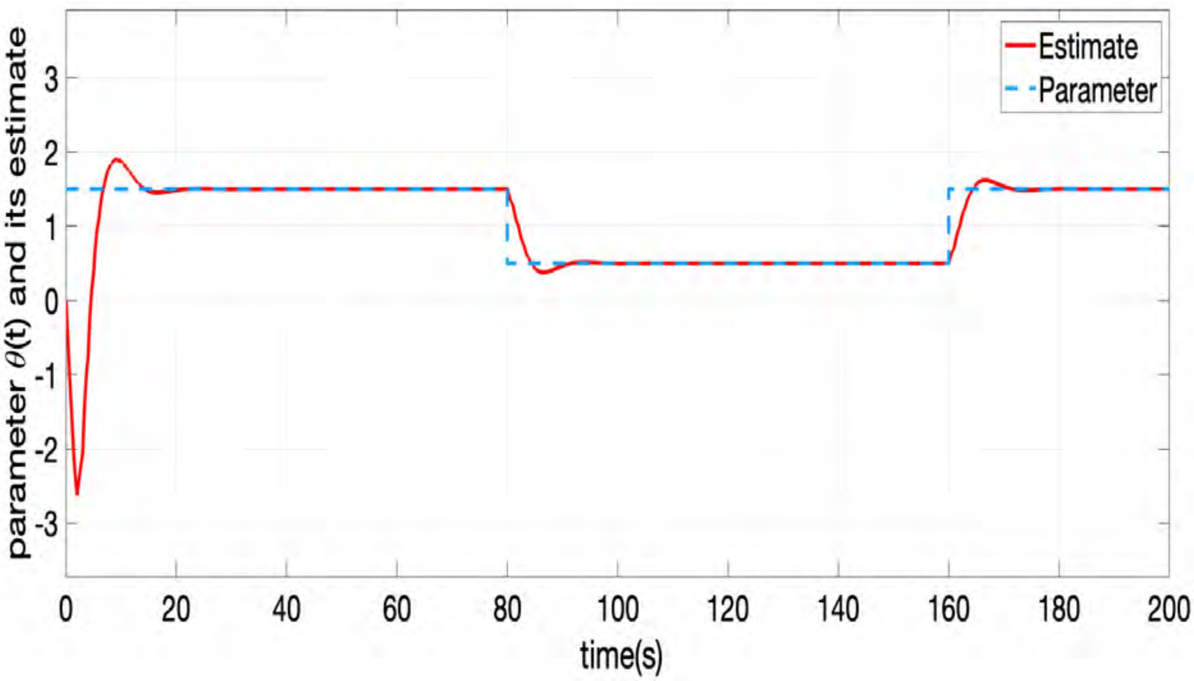

Fig. B.5.6. Convergence of parameter estimate $\theta(t)$.

Figures B.5.2-B.5.5 presents the state estimation error $e_x(t)$ for all the state vector. Figure B.5.6 presents the parameter estimation in relation to its original value. The observer successfully converges to the states of the system. It can be shown that the state estimation errors converge 0 after each change of $\theta(t)$ with a good rate.