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Functional Observers for Descriptor Systems

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Notation

Symbol	**Description**	**Units**
m	Mass	Kg
F	Force	N
J	Moment of inertia	Kgm^2
v	Linear velocity	m/s
g	Gravity acceleration	m/s^2
V	Voltage	V
k	Spring coefficient	N/m
R	Resistor value	Ω
L	Inductance	H
C	Capacitance	F
K	Controller gains	-

Greek letters

Symbol	**Description**	**Units**
θ	Angle of rotation	rad
ω	Angular velocity	rad/s
τ	Torque	Nm

Acronyms

Symbol	**Description**
LTI	Linear Time-Invariant
GDO	Generalized Dynamical Observer
LMI	Linear Matrix Inequality
T-S	Takagi-Sugeno

Abstract

The importance of fault diagnosis and fault-tolerant control systems has been rising. The complex systems used in our daily life require high safety and reliability features, thus fault diagnosis and fault-tolerant systems have an increasing importance in control research.

The reconstruction of the system state has great importance in control theory. Numerous feedback controller require the measurement of all the state for the stabilization of the systems. However, for descriptor systems the conditions for the observer existence are different from standard systems.

This thesis presents a comprehensive study on observer-based control for descriptor systems, with a focus on the application of functional observers and fault-tolerant control methodologies. The research explores the theoretical foundations, design methodologies, and practical implications of functional observers in accurately estimating the states of descriptor systems. The thesis investigates the synthesis of fault-tolerant control strategies to ensure system stability and performance in the presence of sensor and actuator faults. Additionally, the integration of fault estimation techniques within the control framework is explored to achieve simultaneous fault-tolerant control and fault diagnosis capabilities. Theoretical advancements are complemented by extensive simulation-based evaluations, demonstrating the efficacy of the proposed methodologies in various scenarios. Furthermore, the limitations and future research directions are discussed, highlighting the need to address challenges such as actuator and sensor faults, handling nonlinear systems, considering the effects of noise in output measurements, and practical implementation considerations. This research contributes to the advancement of observer-based control for descriptor systems, offering valuable insights and methodologies that can enhance system reliability, performance, and fault tolerance in real-world applications.

Resumen

La importancia del diagnóstico de fallas y de los sistemas de control tolerantes a fallas ha ido en aumento. Los sistemas complejos utilizados en nuestra vida diaria requieren características de alta seguridad y confiabilidad, por lo tanto, el diagnóstico de fallas y los sistemas tolerantes a fallas tienen una importancia creciente en la investigación de control.

La reconstrucción del estado del sistema tiene una gran importancia en la teoría del control. Numerosos controladores de retroalimentación requieren la medición de todo el estado para la estabilización de los sistemas. Sin embargo, para sistemas descriptores, las condiciones para la existencia del observador son diferentes de los sistemas estándar.

Esta tesis presenta un estudio exhaustivo sobre el control basado en observadores para sistemas descriptores, con un enfoque en la aplicación de observadores funcionales y metodologías de control tolerante a fallas. La investigación explora los fundamentos teóricos, metodologías de diseño e implicaciones prácticas de los observadores funcionales para estimar con precisión los estados de sistemas descriptores. La tesis investiga la síntesis de estrategias de control tolerante a fallas para garantizar la estabilidad y el rendimiento del sistema en presencia de fallas en sensores y actuadores. Además, se explora la integración de técnicas de estimación de fallas dentro del marco de control para lograr capacidades de control tolerante a fallas y diagnóstico de fallas simultáneos. Los avances teóricos se complementan con evaluaciones extensas basadas en simulaciones, demostrando la eficacia de las metodologías propuestas en diversos escenarios. Además, se discuten las limitaciones y las direcciones futuras de investigación, destacando la necesidad de abordar desafíos como fallas en actuadores y sensores, manejo de sistemas no lineales, consideración de los efectos del ruido en las mediciones de salida y consideraciones de implementación práctica. Esta investigación contribuye al avance del control basado en observadores para sistemas descriptores, ofreciendo ideas y metodologías valiosas que pueden mejorar la confiabilidad, el rendimiento y la tolerancia a fallas del sistema en aplicaciones del mundo real.

Chapter 1

Introduction

Contents

This chapter serves as a gateway to the exploration undertaken in this thesis, shedding light on the motivations driving the proposed research, examining recent advancements in the field, and elucidating the overarching objectives. Through a comprehensive review, we delve into the background that lays the foundation for the subsequent analyses. The state of the art is scrutinized, with a specific focus on descriptor systems and observers, offering a snapshot of the current landscape. Clear objectives and goals are outlined, providing a roadmap for the ensuing studies. Additionally, the justification for this research is articulated, emphasizing its relevance and potential impact. The chapter concludes by highlighting the original contributions of this work and providing an overview of the document's layout, preparing the reader for a detailed exploration of functional observers and control strategies for descriptor systems.

1.1 Background

State estimation has been an important field of study since the introduction of the concept by D. Luenberger in 1966 [Luenberger, 1966]. In numerous control systems, the full knowledge of the state is needed for the controller design (see Figure 1.1), where $y(t) \in \mathbb{R}^p$ is the measured output, $u(t) \in \mathbb{R}^m$ is the controlled input,$\hat{x}(t) \in \mathbb{R}^n$ is the estimated state, and $u_d(t) \in \mathbb{R}^m$ is the reference input. In most cases the full state is not available for direct measurement thus requiring to estimate the non measurable state.

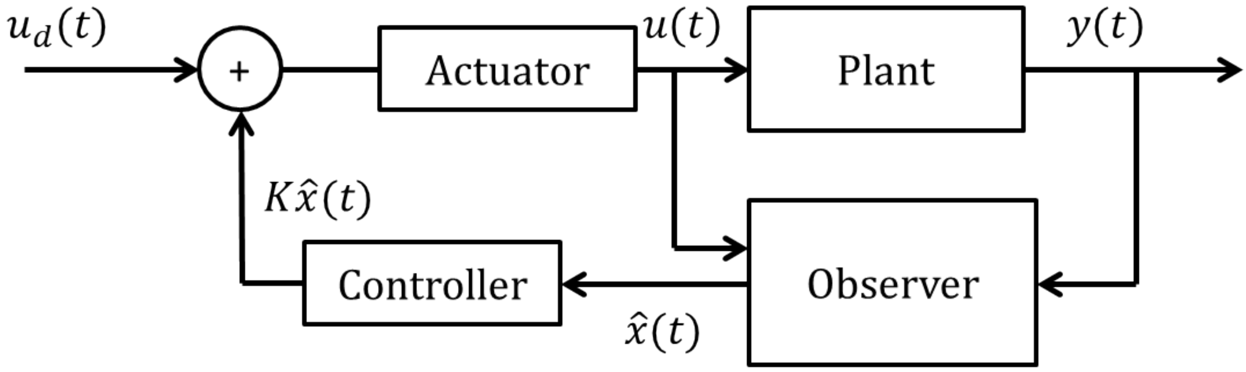


Figure 1.1: Classical observer-based control scheme with $u(t) = K\hat{x}(t) + u_d(t)$

The main purpose of an observer is then to provide the estimation of the state for its use in the control system. This can be achieved in various forms, several observer structures have been proposed since its introduction. An observer can also be used as a powerful tool in control theory. The estimation of the non measurable state is just one of the many applications it can achieve. Depending on the structure of the observer and its design, it can be used to estimate faults, making it a powerful tool for fault diagnostic and fault-tolerant control theory. On the other hand, several structures of observers have been proposed through the years that can be adapted to numerous systems, such as LTI systems, non-linear systems, as well as descriptor and Takagi-Sugeno systems, making them a primary topic in control theory.

As stated above, numerous problems in the field of control theory and fault diagnosis can be solved by using the correct observer, however, this is a research area that can be explored further, new applications and observers can be developed, making them a source of studies for this thesis research.

Moreover, linear descriptor systems have been widely studied, they offer a more natural way to represent the behavior of dynamic systems. We can find dynamic systems in their descriptor form in a variety of fields such as mechanics, chemical processes, electrical and economics [Darouach, 2012]. Depending on the mathematical model of the plant, we can describe an LTI system prone to faults as a linear descriptor system, or we can include unknown inputs or measurement noise into the augmented descriptor system of the plant. Thus, descriptor linear systems offer a powerful tool to solve numerous control problems, using them to represent standard linear systems with unknown disturbances or faults.

1.2 State of the art

The state of the art in the field of functional observers and control for descriptor systems encompasses a diverse range of research and advancements. Over the years, significant progress has been made in developing robust observer designs, fault-tolerant control strategies, and integration techniques for handling uncertainties in real-world applications. Researchers have explored various approaches to enhance the performance, stability, and fault-tolerant behavior of descriptor systems, taking into account complex dynamics, nonlinearities, and practical implementation considerations. By reviewing the existing literature and examining the latest developments, this section provides an overview of the current state of the art in functional observers and control for descriptor systems, laying the foundation for the subsequent exploration of novel methodologies and advancements in this field.

1.2.1 Descriptor systems

Descriptor systems can represent a class of systems which can be described by dynamic equations as well as algebraic equations as seen in [Luenberger, 1978] where the author shows some concepts of descriptor linear discrete-time systems of the form

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1.1}$$

such as the solvability, conditionability and the canonical form for solvable systems. The fact that we can represent systems that include differential and algebraic equations is a powerful tool in control systems. This advantage is the main motivation to design in a descriptor framework several modern observation and control algorithms which can be applied to a wide variety of physical processes such as hydraulic systems like the one shown in [Araujo et al., 2012] where an observer for the system consisting in three interconnected water tanks (see Figure 1.2) is designed.

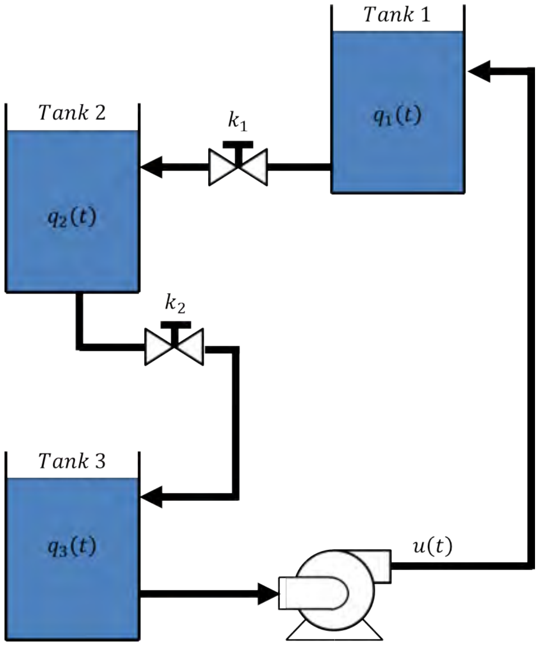


Figure 1.2: Interconnected water tanks used in [Araujo et al., 2012]

The system is modeled in a discrete-time descriptor linear system of the form (1.1). Biomechanical systems have also been studied using a descriptor approach. For example the work of [Guelton et al., 2008], where the human body is modeled as a double pendulum as shown in Figure 1.3, its dynamical behavior is described by a descriptor nonlinear system of the form

$$E(x(t))\dot{x} = A(x(t))x(t) + Bu(t) \tag{1.2}$$

where $x(t)^T = \begin{bmatrix} \theta_1 & \theta_2 & \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix}$.

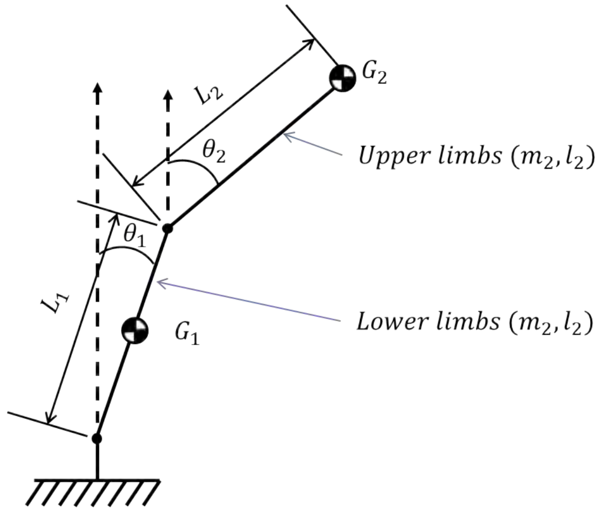


Figure 1.3: Human body stance represented as a double pendulum

In [Lewis, 1986], a survey on descriptor systems is made. Some historical background is given as well as the theories on frequency and time domain for the control of descriptor systems. Some fundamental concepts of descriptor systems, such as regularizability, controllability and observability have been studied in [Cobb, 1984, Zhang, 2006].

In [Kučera and Zagalak, 1988], the authors study the stability and stabilization of descriptor systems. They propose to stabilize a system of the form

$$E\dot{x}(t) = Fx(t) + Gu(t) \tag{1.3}$$

using state feedback with a control law of the form $u(t) = -Kx(t)$ and determining the limits of the state feedback in altering the dynamical behavior of (1.3) . Some restrictions of their work are that the matrix pencil $sE - A$ is assumed to be regular, strictly proper, and regular. The results of the studies are shown using the Kroenecker form.

$$\begin{aligned} \dot{x}_1(t) &= F_1x_1(t) + G_1u(t) \\ E_2\dot{x}_2(t) &= x_2(t) + G_2u(t) \end{aligned} \tag{1.4}$$

The state feedback control has been studied by [Dodig and Stošić, 2009], where a class of singular systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t),$$

where $x(t)$ is the state and $u(t)$ is the input is considered. The control signal is given by

$$u(t) = Kx(t) + v(t),$$

where $v(t)$ is an external input. In this paper, the stabilization of the system is achieved by using matrix pencil theory with the canonical form of the singular system.

Faulty systems can be represented in a descriptor form as seen in [Lan and Patton, 2015], where they represent a descriptor system of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + Ff(t) + Dd(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1.5}$$

where $f(t) \in \mathbb{R}^r$ and $d(t) \in \mathbb{R}^l$ are the fault and the unknown disturbance, respectively. Some restrictions to their work is that the fault is considered constant, i.e. $\dot{f}(t) = 0$. The system (1.5) can be represented as

$$\begin{aligned} \bar{E}\dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}u(t) + \bar{D}d(t) \\ y(t) &= \bar{C}\bar{x}(t) \end{aligned} \tag{1.6}$$

where $\bar{x}^T(t) = \begin{bmatrix} x(t) & f(t) \end{bmatrix}$, $\bar{E} = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}$, $\bar{A} = \begin{bmatrix} A & F \\ 0 & 0 \end{bmatrix}$, $\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$, $\bar{D} = \begin{bmatrix} D \\ 0 \end{bmatrix}$ and $\bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}$. The system (1.6) is then used to design a functional observer that estimates a control law capable to stabilize the system (1.5).

In [Gu and Yang, 2017], a class of nonlinear systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_d d(t) + G\psi(t, x(t), u(t)) \\ y(t) &= Cx(t) + D_f f(t) \end{aligned} \tag{1.7}$$

is represented in a descriptor form as

$$\begin{aligned} \bar{E}\dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}u(t) + \bar{B}_d d(t) + \bar{G}\psi(t, \bar{x}(t), u(t)) + \bar{D}_h h(t) \\ y(t) &= \bar{C}\bar{x}(t) \end{aligned} \tag{1.8}$$

where $\bar{x}^T(t) = \begin{bmatrix} x(t) & h(t) \end{bmatrix}$, $h(t) = D_f f(t)$ $\bar{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{A} = \begin{bmatrix} A & 0 \\ 0 & -I \end{bmatrix}$, $\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$,$\bar{B}_d = \begin{bmatrix} B_d \\ 0 \end{bmatrix}$,$\bar{D}_h = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $\bar{G} = \begin{bmatrix} G \\ 0 \end{bmatrix}$ and $\bar{C} = \begin{bmatrix} C & I \end{bmatrix}$. The equivalent system (1.8) is then used to design an observer that estimates the state of the system and the sensor faults.

The state of the art in descriptor LTI systems encompasses several key areas of research and development. As we conclude our exploration of the existing body of knowledge, it becomes evident that these areas form a dynamic landscape with ongoing advancements and challenges. This comprehensive overview serves as the foundation upon which our research builds, acknowledging the strides made by predecessors while identifying gaps and opportunities for further investigation. Based on this investigation, the state of the art in descriptor LTI systems encompasses several key areas of research and development:

- Modeling and Representation:
  - The state-space representation of descriptor systems, also known as the descriptor form or generalized state-space form, has been widely studied. This representation explicitly incorporates the differential and algebraic equations governing the system's behavior.

  - Various methods have been proposed to transform descriptor systems into regular (non-descriptor) state-space form or to simplify and normalize their equations.
  - System identification techniques for descriptor LTI systems have been explored, allowing the estimation of system parameters from input-output data.

- Stability and Control Analysis:
  - Stability analysis for descriptor systems is a fundamental aspect of control design. Researchers have developed techniques to analyze the stability properties of descriptor systems, including the investigation of stability conditions and the development of stability criteria.
  - Robust control design methodologies have been developed to handle uncertainties in descriptor systems, such as parametric uncertainties, disturbances, and noise.
  - Lyapunov-based methods, such as Lyapunov equations and Lyapunov stability theory, have been extended to address stability analysis and controller synthesis for descriptor LTI systems.

- Observer Design and State Estimation:
  - Observer design for descriptor LTI systems has been a subject of extensive research. Various observer-based techniques have been proposed, including full-order and reduced-order observers, H-infinity observers, and Kalman-like filters.
  - Robust and fault-tolerant observer design methodologies have been developed to handle uncertainties, disturbances, and faults in descriptor systems.
  - Functional observers, a specific class of observers, have gained attention for their ability to estimate both the states and unknown inputs of descriptor systems.

- Control Design and Optimization:
  - Control design for descriptor LTI systems has been investigated, including the design of state feedback controllers, output feedback controllers, and optimal control strategies.
  - Model Predictive Control (MPC) techniques have been adapted and extended to address the control of descriptor systems.
  - Optimization-based control strategies, such as linear programming and quadratic programming, have been employed to handle constraints and achieve desired system performance.

- Applications and Practical Implementations:
  - Descriptor LTI systems find application in diverse fields, including robotics, power systems, chemical processes, and aerospace engineering.

– Real-world case studies and practical implementations of control strategies for descriptor systems have been reported, demonstrating their effectiveness and applicability in industrial settings.

The state of the art in descriptor LTI systems is continuously evolving, with ongoing research focusing on advanced control methodologies, fault diagnosis and accommodation, optimization techniques, and the integration of descriptor models with other control paradigms. This active research area offers exciting opportunities for advancements in control theory and practical applications.

1.2.2 Observers

Observers have been used since the introduction of the concept by D. Luenberger [Luenberger, 1966, Luenberger, 1971]. They are useful as virtual sensors where part of the state is not available for direct measurement by physical sensors. State estimators for linear time-invariant systems have been widely studied [Ali et al., 2015]. On the other hand, the full state is not always needed to be estimated since we only need the state that is not available for direct measurement [Trinh and Fernando, 2011]. This can be achieved with a functional observer which estimates a linear combination of the state [Fernando et al., 2010, Fernando et al., 2011], this is useful in control theory since we can estimate a control signal capable of stabilize an unstable system.

It is known that conventional proportional observers are sensitive to parametric uncertainties and disturbances, this leads to an increase in the estimation error in steady state. Proportional-Integral observers have been proposed to solve this problem, as shown by [Shafai and Carroll, 1985] who designed an adaptive Luenberger-based observer to estimate the state of a dynamical system with unknown disturbances of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Dd(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1.9}$$

In order to estimate the state, and the unknown disturbance, an adaptive observer of the form

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + D\hat{d}(t) + K_p(y(t) - C\hat{x}(t)) \\ \dot{\hat{d}}(t) &= K_I(y(t) - C\hat{x}(t)) \end{aligned} \tag{1.10}$$

is proposed.

The observer (1.10) is then used with a quadratic regulator to stabilize a linear dynamical system in [Shafai et al., 2002].

There have been numerous authors that propose methods to design functional observers for Linear Time-Invariant (LTI) systems as in [Zhang, 1987]. Here, the authors propose the design of a functional observer using the frequency domain analysis. In [Yao et al., 1997], a functional observer design for time delayed systems is proposed using the frequency domain analysis. The stabilization of standard linear systems has been studied in [Aldeen and Trinh, 1999] where the authors propose a functional observer to stabilize an unstable standard linear system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1.11}$$

They propose a control law $u(t) = Kx(t)$ and decompose the feedback gain matrix as $K = NT + WC$. The estimated control law becomes then $\hat{u}(t) = Pz(t) + Qy(t)$, where $z(t) = Tx(t)$ is an auxiliary state of the observer. By proposing $\dot{z}(t) = Nz(t) + TBu(t) + Fy(t)$, the problem of the observer design is reduced to determine the gain matrices P, Q, N, T and F.

State estimation for descriptor systems have been studied by [Darouach and Boutayeb, 1995, Darouach et al., 1996,Darouach, 2000,Darouach, 2001,Müller and Hou, 1991] where the authors propose different structures of observers for descriptor linear systems of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1.12}$$

More recently, [Darouach, 2012] shows a more generalized structure of functional observer for descriptor systems. A functional observer of the form

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + F \begin{bmatrix} E^{\perp}Bu(t) \\ y(t) \end{bmatrix} + Hu(t) \\ \hat{z}(t) &= P\zeta(t) + Q \begin{bmatrix} E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \end{aligned} \tag{1.13}$$

In [Osorio-Gordillo et al., 2016], the authors propose a design method for Generalized Dynamic Observers (GDO)for descriptor LTI systems of the form (1.12) .The observer structure is constructed as it follows

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + Jv(t) + F \begin{bmatrix} E^{\perp}Bu(t) \\ y(t) \end{bmatrix} + Hu(t) \\ \dot{v}(t) &= S\zeta(t) + Gv(t) + M \begin{bmatrix} E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \\ \hat{z}(t) &= P\zeta(t) + Q \begin{bmatrix} E^{\perp}Bu(t) \\ y(t) \end{bmatrix}. \end{aligned} \tag{1.14}$$

In [Osorio-Gordillo et al., 2015], the observer is used to estimate the state of a descriptor system with unknown disturbances. The observer is designed under the H_{∞} optimal constraints. Finite-time convergence, is not restricted to sampling and discrete time techniques, thus, finite-time convergence is an appealing topic of study [Engel and Kreisselmeier, 2002]. This is specially important in fault estimation because a large number of practical control system require short response constraints. Designing an observer capable to reach correct convergence within a finite-time constraint can be a powerful tool for fault diagnosis.

Observers play a crucial role in control systems by estimating the unmeasured states of a system based on available measurements. Over the years, various observer designs have been proposed, including well-known techniques such as Luenberger observers, Kalman filters, and extended Kalman filters. Each of these designs has its own set of advantages and limitations, and researchers have focused on developing observer methodologies that can handle nonlinearities, uncertainties, and disturbances encountered in real-world systems. Advanced observer techniques, such as sliding mode observers, high-gain observers, and adaptive observers, have been extensively investigated to improve estimation accuracy and robustness. Moreover, observer-based control strategies, including state feedback control and output feedback control, have been developed to achieve desired system performance based on the estimated states.

Functional observers have emerged as a specific class of observers that estimate not only the states but also the unknown inputs or disturbances affecting a system. They have garnered significant attention for their capability to estimate unmeasured variables or disturbances that directly impact the system dynamics. Researchers have focused on the design and analysis of functional observers, delving into their theoretical foundations and exploring their application to a wide range of control problems. Various types of functional observers have been proposed, such as proportional-integral (PI) observers, sliding mode observers, and high-gain observers. Furthermore, the integration of functional observers with control strategies, including fault-tolerant control and adaptive control, has been extensively investigated to enhance system performance and robustness in the face of uncertainties. The development of robust and fault-tolerant functional observers has also been a subject of interest, aiming to handle uncertainties, disturbances, and faults in real-world systems. Practical implementations and case studies have demonstrated the effectiveness of functional observers in diverse applications, ranging from power systems to robotics and automotive control.

In summary, the state of the art in observers and functional observers is a dynamic field of research that continues to evolve. Ongoing studies are focused on advanced observer designs, robustness analysis, fault diagnosis and accommodation, nonlinear and adaptive observer techniques, and the integration of functional observers with control strategies. These research efforts open up new possibilities for advancements in state estimation, system monitoring, and control across various applications, creating opportunities to improve the performance and robustness of real-world systems.

1.3 Objectives

1.3.1 General objective

To obtain the estimation of a linear combination of the system states by designing a functional observer. The linear function can be considered as the states of the system or as a state feedback controller.

1.3.2 Specific objectives

- To establish the theoretical foundations and mathematical principles underlying functional observers for accurate state estimation.
- To develop methodologies for designing functional observers tailored to descriptor systems.
- Investigate the necessary conditions for the existence of functional observers and propose design guidelines.
- To explore fault detection and fault diagnosis techniques suitable for descriptor systems.
- To develop strategies for fault accommodation and fault-tolerant control based on the information provided by the functional observers.
- To conduct extensive simulations to assess the performance of the developed methodologies in various scenarios.

1.4 Justification

In numerous control systems, the complete state of the system is generally not needed. State feedback, for example, require a linear combination of the state. Functional observers are useful since they can be used to estimate a linear combination of the state of a system. This advantage is used to design reduced order observers for the state and fault estimation in control systems. Functional observers can be used to design reduced order observer-based control systems and fault-tolerant control systems.

This research project stems from the growing demand for advanced control techniques that can address the complex dynamics and uncertainties encountered in practical systems. The existing literature reveals the need for robust state estimation methods and fault-tolerant control strategies that can enhance system performance, reliability, and adaptability. By undertaking this research project, we aim to bridge the existing gaps and contribute to the field by developing innovative methodologies that can accurately estimate system states, accommodate faults, and enable robust and efficient control in descriptor systems. The outcomes of this research project have the potential to make significant contributions to various industrial domains and enhance the understanding and practical applicability of functional observers and control techniques for descriptor systems.

1.5 Originality

Contrary to previous works, this thesis takes advantage of the generalized form of state observers to propose a form and design of a functional observer for descriptor systems. The results are then extended to applications such that finite-time estimation and systems prone to parametric uncertainties.

While significant progress has been made in control theory, this research project introduces an innovative methodology and approach that address specific challenges and fill critical gaps in the current literature. By combining advanced techniques and leveraging recent developments, this project offers a fresh perspective and novel solutions for accurate state estimation, fault-tolerant control, and system adaptability in descriptor systems. The originality of this research lies in the development of new algorithms, theoretical frameworks, and integration strategies that go beyond the existing approaches, providing novel insights and pushing the boundaries of knowledge in the field. By presenting original contributions, this research project aims to advance the scientific community's understanding of functional observers and control for descriptor systems, offering unique and valuable perspectives that have the potential to significantly enhance the performance and robustness of these systems.

1.6 Document layout

This document is divided in four parts:

- **Chapter I.** This chapter lays the groundwork by elucidating the objectives and goals of the thesis proposal. It delves into the justification that propelled the exploration of the chosen topic, providing a comprehensive overview of the motivations behind the research.
- **Chapter II.** Titled "Theoretical Framework," this chapter serves as a comprehensive exploration of fundamental concepts and the current state of the art related to the topics under study. It establishes a theoretical foundation essential for the subsequent analyses and discussions.
- **Chapter III.** In this chapter, titled "Functional Observers for LTI Systems," the focus is on the design of Functional Observers tailored for LTI descriptor systems. The chapter extends its scope to encompass systems with unknown inputs, and culminates in an examination of state and fault estimation for Takagi-Sugeno systems.
- **Chapter IV.** Titled "Application of Functional Observers," this chapter explores the potential practical implementations of functional observers. It covers observer-based control for descriptor LTI systems, its utilization in fault detection and estimation, and introduces the concept of observer-based fault-tolerant control.
- **Chapter V.** The final chapter, "Conclusion," consolidates the findings and insights gleaned throughout the research. It encapsulates general conclusions drawn from the study and outlines potential avenues for future research, providing a comprehensive closure to the thesis.

Chapter 2

Theoretical Framework

Contents

This chapter serves as the foundation for the research conducted in this thesis on functional observers and control for descriptor systems. It presents a comprehensive overview of the theoretical underpinnings and key concepts essential for understanding the subsequent chapters. By delving into the fundamental principles and mathematical tools used in the analysis and design of functional observers and control strategies, this chapter establishes a solid theoretical framework. Through this comprehensive theoretical framework, this chapter sets the stage for the subsequent chapters to delve into more specific aspects of functional observers and control for descriptor systems, providing the necessary theoretical groundwork to advance the understanding and application of these techniques in practical scenarios.

It begins with a detailed examination of descriptor systems, unraveling their intricacies and delving into examples that illuminate their relevance in practical applications. Properties intrinsic to descriptor systems are then scrutinized, shedding light on key characteristics that form the basis for subsequent discussions. Moving forward, the exploration extends to Linear Matrix Inequality (LMI) regions, with a particular focus on D-stability, providing a mathematical lens through which system stability can be assessed.

The chapter further advances into Takagi-Sugeno systems, exploring their unique characteristics and applicability in handling complex, nonlinear systems. Finally, the theoretical framework encompasses functional observers, elucidating their role in system observation and control. By establishing a firm theoretical groundwork, this chapter propels the subsequent sections to delve into more nuanced aspects of functional observers and control for descriptor systems. This groundwork is essential for advancing the comprehension and practical application of these techniques in real-world scenarios, providing a comprehensive toolkit for researchers and practitioners alike.

2.1 Descriptor systems

Descriptor systems are a class of systems that include the dynamic and static behavior of a plant. Mathematically, they are represented by a set of dynamic and algebraic equations. Descriptor systems are a powerful modeling tool since they can describe behaviors that standard dynamical systems cannot.

Nonlinear descriptor systems can be represented by the following set of equations:

$$\begin{aligned} E\dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t)) \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input and $y(t) \in \mathbb{R}^p$ is the measured output. $f(\cdot)$ and $g(\cdot)$ are nonlinear continuous and infinitely differentiable functions. E is a singular constant matrix with appropriate dimensions. If system (2.1) is linealized, the result in continuous case is as it follows.

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{2.2}$$

Linear descriptor systems have features that does not exist or are different than standard LTI systems. Impulsive behavior and impulsive modes are important features of descriptor systems and need to be solved in order to design a control system. Concepts as regularity, reachability, controllability and observability will be presented in this section the concepts are extracted form [Duan, 2010] and [Dai, 1989].

2.1.1 Examples

Descriptor modeling can be used to describe the interaction between different systems, for example, in 2.1, we can see the electric circuit of an electrical motor and its interaction in a mechanical rotary system.

Electrical motor

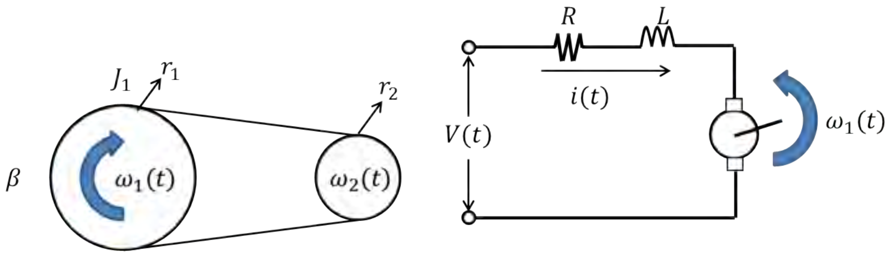


Figure 2.1: Electrical - Mechanical system

From Figure 2.1 we can derive three equations: a dynamic equation from the mechanical system, another dynamical equation form the motor circuit, and an algebraic equation that couples both systems.

$$\begin{aligned} J\dot{\omega}_1(t) &= -\beta\omega_1(t) + K_T i(t) \\ L\dot{i}(t) &= -K_b\omega_1(t) - Ri(t) + V(t) \\ r_1\omega_1(t) - r_2\omega_2(t) &= 0 \end{aligned} \tag{2.3}$$

$$\begin{bmatrix} J & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\omega}_1(t) \\ \dot{i}(t) \\ \dot{\omega}_2(t) \end{bmatrix} = \begin{bmatrix} -\beta & K_T & 0 \\ -K_b & R & 0 \\ r_1 & 0 & -r_2 \end{bmatrix} \begin{bmatrix} \omega_1(t) \\ i(t) \\ \omega_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} V(t) \tag{2.4}$$

Linearized single-link robot arm

In this case, we present a single-link robot arm which is connected to an electrical motor (as it can be seen in Figure 2.2).

The mathematical model that represents the system is:

$$\begin{aligned} J\ddot{\theta}(t) &= -mgl_c \sin(\theta(t)) - \beta\dot{\theta}(t) + lF_l + d(t) \\ F_l &= C_s\omega(t)^2 \end{aligned} \tag{2.5}$$

where:

- $\theta(t)$ is the position of the arm in rad.
- $F_l(t)$ is the lift force in N.
- $d(t)$ is an unknown input.
- J is the moment of inertia of the pendulum in Kg/m^2.
- m is the mass of the arm in Kg.

- l_c is the distance from the pivot point to the center of gravity in m.
- β is the viscous coefficient of friction of the pivot point in Nms/rad.
- l is the length of the arm in m.
- C_s is a lift constant in $Kg * m$.
- g is the acceleration of gravity in m/s^2-

Takagi-Sugeno model

The mathematical model of single link robot arm (see Figure 2.2).

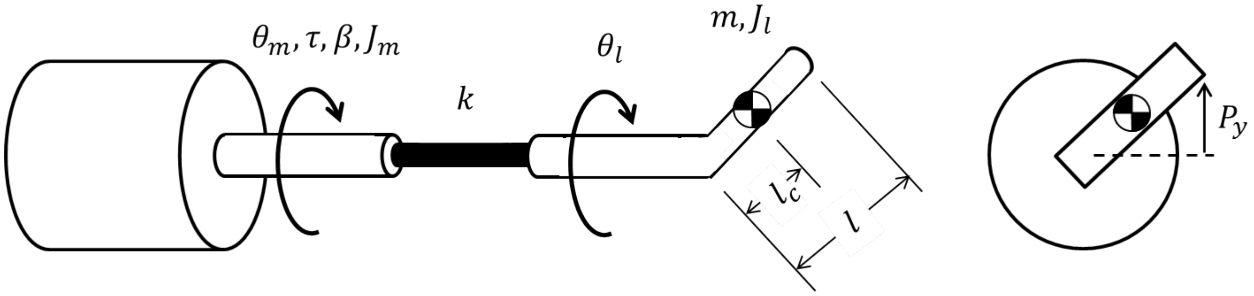


Figure 2.2: Single link robot arm

The model was selected from [Chakrabarty et al., 2017] with the following nonlinear model:

$$\begin{aligned} J_m\ddot{\theta}_m(t) &= -\beta\dot{\theta}_m(t) - k(\theta_m(t) - \theta_l(t)) + K_\tau u(t) \\ J_l\ddot{\theta}_l(t) &= k(\theta_m(t) - \theta_l(t)) + l_c mg \sin\theta_l(t) \end{aligned} \quad (2.6)$$

where:

- θ_m is the angular position of the motor shaft.
- θ_l is the angular position of the link.
- J_m and J_l are the moment of inertia of the motor shaft and the link, respectively.
- β is the viscous coefficient of friction of the motor.
- k is the spring rigidity coefficient.
- m is the mass of the link.
- l_c is the distance of the center of mass.
- l is the length of the link.

- K_τ is the constant of the motor.

Taking as the states $x_1(t) = \theta_m(t)$, $x_2(t) = \dot{\theta}_m(t)$, $x_3(t) = \theta_l(t)$, $x_4(t) = \dot{\theta}_l(t)$ and $x_5(t) = d_y(t)$, we can represent (2.6) as it follows:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -\frac{\beta}{J_m}x_2(t) - \frac{k}{J_m}x_1(t) + \frac{k}{J_m}x_3(t) + \frac{K_\tau}{J_m}u(t) \\
\dot{x}_3(t) &= x_4(t) \\
\dot{x}_4(t) &= \frac{k}{J_l}x_1(t) - \frac{k}{J_l}x_3(t) + \frac{l_c mg}{J_l}\sin x_3(t) \\
0 &= l \sin x_3(t) - x_5(t)
\end{aligned} \tag{2.7}$$

its representation as a nonlinear descriptor system is

$$E\dot{x} = f(x,t) + Bu(t) \tag{2.8}$$

where $E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ \frac{K_\tau}{J_m} \\ 0 \\ 0 \\ 0 \end{bmatrix}$,and $f(x,t) = \begin{bmatrix} x_2(t) \\ -\frac{\beta}{J_m}x_2(t) - \frac{k}{J_m}x_1(t) + \frac{k}{J_m}x_3(t) \\ x_4(t) \\ \frac{k}{J_l}x_1(t) - \frac{k}{J_l}x_3(t) + \frac{l_c mg}{J_l}\sin x_3(t) \\ l\sin x_3(t) - x_6(t) \end{bmatrix}$.

We chose as the scheduling variable $\rho(t) = \frac{\sin x_3(t)}{x_3(t)}$, so the system (2.7) can be represented:

$$\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{k}{J_m} & -\frac{\beta}{J_m} & \frac{k}{J_m} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{k}{J_l} & 0 & \frac{1}{J_m}\left(k - l_c mg\rho(t)\right) & 0 & 0 \\ 0 & 0 & l\rho(t) & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{K_\tau}{J_m} \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} x(t)
\end{aligned} \tag{2.9}$$

The weighting functions for the T-S systems are:

$$w_1(t) = \frac{\bar{\rho} - \rho(t)}{\bar{\rho} - \underline{\rho}} \tag{2.10}$$

$$w_2(t) = 1 - w_1(t) \tag{2.11}$$

The Takagi-Sugeno system can be then represented as it follows:

$$\begin{aligned}
E\dot{x}(t) &= \sum_{i=1}^{2} w_i(t)\left(A_i x(t) + Bu(t)\right) \\
y(t) &= Cx(t)
\end{aligned} \tag{2.12}$$

with

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -30 & -5 & 30 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 3.75 & 0 & -8.33 & 0 & 0 \\ 0 & 0 & 0.18 & 0 & -1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -30 & -5 & 30 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 3.75 & 0 & -2.77 & 0 & 0 \\ 0 & 0 & -0.04 & 0 & -1 \end{bmatrix}$$

2.1.2 Descriptor systems properties

In this section, the fundamental properties of the descriptor systems are discussed.

Regularity

For a standard LTI system there exists always an unique solution, this is however not always the case for descriptor systems. A solution for a descriptor system can be non existent or can have multiple solutions. The conditions leading to the existence and uniqueness of a solution result in the concept of regularity. Regularity is a very important property of a descriptor linear system since it guarantees the existence and uniqueness of a solution.

Definition 2.1.1 *A system of the form* (2.2) *is said to be regular if there exists a scalar* $\gamma \in \mathbb{C}$ *such that the following condition holds.*

$$det(\gamma E - A) \neq 0 \tag{2.13}$$

Equation (2.13) *is equivalent to say that the polynomial equation* $det(sE - A)$ *is not equal to zero. In this case we say that the pair* (E, A) *is regular.*

Regularity also introduces the concept of regularizability which is an important topic in control theory since its goal is to present the conditions for the existence of a controller that guarantees that the closed loop system is regular.

Stability

Stability is a key factor in any control system. It is well known that a practical control system should be stable, otherwise, it can lead to the incorrect behavior of the system or even its destruction.

We consider now a class of systems of the form

$$E\dot{x}(t) = Ax(t), \quad x(0) = x_0 \tag{2.14}$$

We assume that (2.14) is regular since it has an unique solution and makes the definition easier. We know that the system is stable if $\lim_{t\to\infty} x(t) = 0$.

Definition 2.1.2 *A descriptor linear system is said to be stable if and only if*

$$\sigma(sE - A) \subset \mathbb{C} = \{s|s \in \mathbb{C},\ Re(s) \leq 0\} \tag{2.15}$$

where $\sigma(sE - A)$ denotes the poles of the pair (E, A).

Descriptor system transformation

Regular descriptor systems can be divided in two subsystems, a slow system and a fast system. The slow system is a set of ordinary differential equations with an unique solution and its response is determined by the initial conditions and the input signal $u(t)$, the fast system can have impulsive behaviors. Both systems, the slow and the fast can be obtained using a singular value decomposition which tells us that there exist two matrices P and Q such that.

$$\begin{aligned} \tilde{E} &= PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} \\ \tilde{A} &= PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \\ \tilde{B} &= PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \end{aligned} \tag{2.16}$$

where $N \in \mathbb{R}^{n_2 \times n_2}$ is a nilpotent matrix with nilpotent index h. The system (2.2) is equivalent to the system

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t) \\ N\dot{x}_2 &= x_2(t) + B_2 u(t) \end{aligned} \tag{2.17}$$

Impulsive-free behavior

Impulsive behaviors are usually undesired in practical applications. The impulsive term and its derivatives can saturate the state or even damage the system. Impulse-free descriptor systems are thus more practical for control applications. There exists various methods for eliminating the impulsive behavior, however, for these thesis studies, all descriptor systems presented are assumed to be impulse-free.

A regular descriptor system is said to be impulse-free if and only if one of the following conditions is true:

1. The nilpotent matrix N in its fast subsystem is a zero matrix.
2. $deg(det(\lambda E - A)) = rank(E)$
3. The system has $rank(E)$ finite poles.

Admissibility

A descriptor system is said to be admissible if it is both: stable and impulse free. The admissibility of the descriptor linear system of the form

$$E\dot{x}(t) = Ax(t) \tag{2.18}$$

is also related to a type of generalized Lyapunov matrix equations, but in the following form:

$$E^T XA + A^T XE = -E^T YE \tag{2.19}$$

where $Y = Y^T < 0$.

The admissibility of a descriptor system of the form (2.18) is guven by the following equivalent statements:

- *If there exist* $X = X^T \geq 0$*, and* $Y = Y^T > 0$ *such that LMI* (2.19)*, then system* (2.18) *is admissible.*

- *If the pair* (E, A) *is admissible, then for every* $y > 0$ *there exist a matrix* $X > 0$ *that satisfies* (2.19)

If we consider the system 2.19, then the the following statements are true:

- The pair (E, A) is said to be regular if and only if $det(\lambda E - A)$ is not identically 0.

- The pair (E, A) is said to be impulse-free if $deg(det(\lambda E - A)) = rank(E)$

- The pair (E, A) is said to be stable if all the roots in $(\lambda E - A) = 0$ have negative real parts.

- The pair (E, A) is said to be admissible if it is regular, impulse-free, and stable.

The following theorem gives the necessary conditions for system (2.19) to be admissible.

Theorem 2.1.3 *The pair* (E, A) *is said to be admissible of and only if there exists a matrix* P*, such that*

$$\begin{aligned} EP_c^T = P_c^T E \geq 0 \\ P_c^T A + A^T P_c < 0 \end{aligned} \tag{2.20}$$

or

$$\begin{aligned} EP_c = P_c^T E^T \geq 0 \\ AP_c + P_c^T A^T < 0 \end{aligned} \tag{2.21}$$

Controllability and observability

In general, descriptor systems are fixed into algebraic constraints, the state of a dynamical system is then bounded. A vector $w(t) \in \mathbb{R}^n$ is said to be reachable if there exists an initial condition $x(0) = x_0$ and an admissible input $u(t) \in \mathbb{R}^m$ such that the state satisfies $x(t_1) = w(t)$ in a finite time $t_1 > 0$. The set that contains all the reachable vectors is called reachable set and it is defined as

$$\mathscr{R}_t[x_{10}] := \{w \mid \exists u(t) \in \mathbb{R}^m : x(t, u, x_{10}) = w\} \tag{2.22}$$

the reachable set $\mathscr{R}_t[x_{10}]$ is the set of reachable vectors at a time t form an initial sate $x_1(0) = x_{10}$.

Reachability is an important concept since the controllability and observability of a descriptor system are associated with its reachability. As mentioned earlier, controllabilty and observability are important features of a system. In standard systems, for example, the poles of a feedback system can be arbitrarily assigned if and only if the system is controllable, similarly, the poles of a full order observer for a standard system can be arbitrarily assigned if and only if the system is observable.

On the other hand, there exist multiple definitions of controllability and observability for descriptor linear systems. These definitions are based on the controllability and observability of the slow and the fast system in a descriptor system of the form (2.17).

C-Controllability A descriptor system of the form (2.17) is said to be *completely controllable*, or C-Controllable, if there exists a finite time $t_1 > 0$, an initial value $x_0 \in \mathbb{R}^n$ and an objective vector $w \in \mathbb{R}^n$ for which there is a control input $u(t)$ that satisfies $x(t_1, u, x_0) = w$. This means that for any initial condition x_0 we can find a control input $u(t)$ such that the state of the system is driven to any desired position in $\mathbb{R}^n$ in a finite time. This leads to the following definition.

Definition 2.1.4 *A regular descriptor system of the form* (2.17) *is C-Controllable if and only if*

$$\mathscr{R}_t[0] = \mathbb{R}^n, \ \forall t > 0 \tag{2.23}$$

For a regular descriptor system, it is necessary that both, the slow and the fast system are C-Controllable.

R-Controllability A regular descriptor system is said to be *controllable in a reachable set*, or R-Controllable, if for any initial condition x_0 it is possible to reach any state within the set $\mathscr{R}$ in a finite time $t_1 > 0$ with a control input $u(t)$.

Generally the descriptor systems are limited by algebraic constraints, this means that, in general, descriptor sytems cannot be C-Controllable. The following definition applies

Definition 2.1.5 *A regular descriptor system of the form* (2.17) *is said to be R-Controllable if*

$$\mathscr{R}_t[0] = \mathscr{R}, \ \forall t > 0 \tag{2.24}$$

where $\mathscr{R}_t$ *represents the reachable set for an initial condition* 0.

I-Controllability The concepts of C-Controllability and R-Controllability are only concerned in the steady state behavior of a descriptor system, yet, it is known that impulsive behaviors may exist at $t = 0$ in descriptor systems. Impulsive behavior can lead to malfunction of the control system, so it is necessary to eliminate the unwanted impulsive behaviors. This introduces the concept of I-Controllability, this is the ability to cancel all impulsive behaviors using an admissible control input $u(t)$. This can be reached by making the response of system finite the moment of the impulse occurs. We propose now the following definition.

Definition 2.1.6 *A regular descriptor system of the form* (2.17) *is I-Controllable if and only of for any vector* $x_{20} \in \mathbb{R}^n$*, there exists an admissible control input* $u(t)$ *such that*

$$Nx_{20} + \begin{bmatrix} NB_2 + N^2B_2 + \cdots + N^{h-1}B_2 \end{bmatrix} \begin{bmatrix} u(0) \\ \dot{u}(0) \\ \vdots \\ u^{(h-1)}(0) \end{bmatrix} = 0 \tag{2.25}$$

S-Controllability The descriptor linear system (2.17) is said to be *strongly controllable* or S-Controllable if it satisfies the conditions of R-Controllability and I-Controllability.

The relations between the different concepts of controllability are described by the diagram of [Duan, 2010]:

$$\text{C-Controllability} \Rightarrow \text{S-Controllability} \Leftrightarrow \text{R-Controllability} + \text{I-Contollability}$$

C-Observability We define observability as the ability of reconstruct the state of the system from the inputs and output measurements. Like the controllability concept, observability have multiple definitions for descriptor systems based on the ability to reconstruct the slow and the fast system (based on descriptor systems of the form (2.17)). For this section we present a descriptor system of the form

$$E\dot{x}(t) = Ax(t) \tag{2.26a}$$

$$y(t) = Cx(t) \tag{2.26b}$$

using the single value decomposition, the descriptor system (2.26) can be represented in the form

$$\begin{aligned} \dot{x}_1(t) &= A_1x_1(t) \\ y_1(t) &= C_1x_1(t) \end{aligned} \tag{2.27a}$$

$$\begin{aligned} N\dot{x}_2(t) &= x_2(t) \\ y_2(t) &= C_2x_2(t) \end{aligned} \tag{2.27b}$$

where $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$. The pair of transformation is assumed to be (P, Q) this gives us that $P^{-1}x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $QEP = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}$, $QAP = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}$ and $CP = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$.

We say then that a regular descriptor system is *completely observable* or C-Observable if the initial condition $x(0)$ can be uniquely determined by $u(t)$ and $y(t)$. The observability of a descriptor linear system can be determined by its observability matrices $\mathcal{Q}_0$ for systems of the form (2.27) we have two observability matrices, one for the fast system and one the the slow one. These are defined as

$$\mathcal{Q}_0[A_1, C_1] = \begin{bmatrix} C_1 \\ C_1 A_1 \\ \vdots \\ C_1 A_1^{n-1} \end{bmatrix} \tag{2.28}$$

and

$$\mathcal{Q}_0[N, C_2] = \begin{bmatrix} C_2 \\ C_2 N \\ \vdots \\ C_2 N^{n-1} \end{bmatrix} \tag{2.29}$$

This leads to the following definition

Definition 2.1.7 *The regular descriptor system is C-Observable if and only if both, the fast and the slow systems are C-Observable. They can be determined by*

1. *The slow system is C-Observable if and only if*

$$rank(\mathcal{Q}_0[A_1, C_1]) = n_1$$

2. *The fast system is C-Observable if and only if*

$$rank(\mathcal{Q}_0[N, C_2]) = n_2$$

R-Observability The system (2.26) is said to be *observable within the reachable set* or R-Observable if any state in the reachable set can be can be uniquely determined by $y(t)$ and $u(t)$.

I-Observability The regular descriptor system (2.26) is *impulse observable* or I-Observable if the impulse behavior in the state response $x(t)$ can be determined uniquely by the impulse behavior of the output and the jump behavior in the input.

S-Observability As for the controllability, a regular descriptor system (2.26) is *strongly observable* if it is both R-Observable and I-Observable.

Partial Observability Consider the following consistent linear equation system of unknowns x and z.

$$\mathcal{A}x = b \tag{2.30a}$$

$$z = \mathcal{L}x \tag{2.30b}$$

Partial observability concerns the capability to reconstruct a function of the state of a system based on the measurements of its outputs. This can be deduced based on the following lemmas.

Lemma 1 *For any two matrices $\mathcal{A}$ and $\mathcal{L}$, $Im(\mathcal{L}) \subset Im(\mathcal{A})$ if and only if there exist a matrix Φ such that $\mathcal{L} = \Phi\mathcal{A}$.*

The proof of this lemma is well known in linear algebra theory.

Lemma 2 *The following statements are equivalent.*

1. *The functional z is uniquely determined by* (2.30a)

2. $Im(\mathcal{L}) \subset Im(\mathcal{A})$

3. $rank \begin{bmatrix} \mathcal{L} \\ \mathcal{A} \end{bmatrix} = rank\, \mathcal{A}$

Proof 1 *The proof of this Lemma can be found in [Darouach, 2012]*

2.2 LMI regions

A subset of the complex plane is called an n-order LMI region if there exists a symmetric matrix $\mathbb{M} \in \mathbb{R}^n \times n$ and a matrix $\mathbb{N} \in \mathbb{R}^n \times n$ such that:

$$R = \left\{ z \in C : f_R(z) = \mathbb{M} + \mathbb{N}z + \mathbb{N}^T z < 0 \right\} \tag{2.31}$$

It can be seen that the characteristic function f_R of the complex variable z takes values in the space of n-order Hermithian matrices and that the LMI region is symmetric with respect to the x axis since $f_R^T(z) = f_R(z)$. If all the poles of a system are in the subset R. It is said that the system is R-stable. This can be used to obtain a desired dynamical behavior of a system since we can use control feedback to place the poles of said system in a desired LMI region [Chilali and Gahinet, 1996].

2.2.1 $\mathcal{D}$-stability

A system is said to be $\mathcal{D}$-stable if all of its eigenvalues are inside a disk region $\mathcal{D}(\alpha, \tau)$, centered in $(\alpha, 0)$ and with radius τ (see Fig. 2.3).

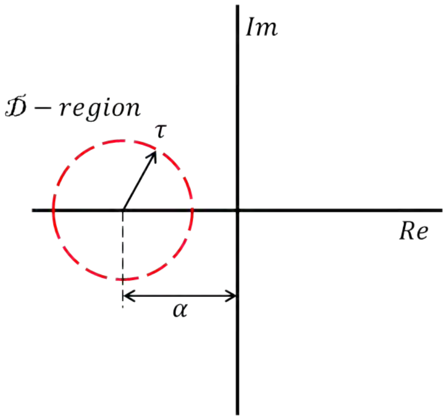


Figure 2.3: $\mathcal{D}$-stability region

Lemma 3 *[Zhang et al., 2016] The eigenvalues of two given matrices $\mathcal{A} \in \mathbb{R}^{n\times n}$, belong to the disk region $\mathcal{D}(\alpha, \tau)$ with center $\alpha + 0i$ and radius τ if and only if there exists a symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{n\times n}$ such that the following condition holds:*

$$\begin{bmatrix} -\mathcal{P} & \mathcal{P}\mathcal{A} - \alpha\mathcal{P} \\ * & -\tau^2\mathcal{P} \end{bmatrix} < 0 \tag{2.32}$$

2.3 Takagi-Sugeno systems

A Takagi-Sugeno system, also known as a Takagi-Sugeno-Kang (TSK) model, is a type of fuzzy model used in control systems. It is named after its creators, Michio Takagi, Mamoru Sugeno, and Tomomitsu Kang. A Takagi-Sugeno system is a mathematical representation of a nonlinear system using a set of fuzzy rules.

In a Takagi-Sugeno system, the overall system behavior is modeled as a collection of local linear models, where each local model corresponds to a fuzzy rule. The fuzzy rules consist of linguistic variables and fuzzy sets that describe the system's inputs and outputs. These linguistic variables are associated with membership functions that quantify the degree of membership of a given input or output to a specific fuzzy set.

Each fuzzy rule in a Takagi-Sugeno system defines a linear relationship between the system inputs and the corresponding output. These linear relationships are often defined using weighted averages, where the weights are determined by the degree of membership of the input variables in the fuzzy sets. The overall output of the Takagi-Sugeno system is then obtained by aggregating the outputs of all the fuzzy rules, typically using a weighted sum or a weighted average.

Takagi-Sugeno systems have gained popularity due to their ability to approximate nonlinear systems using a combination of local linear models. They provide a flexible and interpretable framework for modeling complex systems and designing control strategies. The parameters of the Takagi-Sugeno model, such as the fuzzy sets and the linear coefficients, can be determined through identification techniques, including data-driven approaches and expert knowledge.

Overall, Takagi-Sugeno systems offer a powerful tool for modeling and controlling nonlinear systems, enabling the design of robust and adaptive control strategies based on fuzzy logic and approximate reasoning.

2.4 Functional observers

The problem of reconstructing the state of a system is one of the most important topics in control theory, since the works of D. Luenberger in 1966 where he presents an observer structure for Linear Time-Invariant (LTI) systems of the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ z(t) &= Lx(t) \end{aligned} \tag{2.33}$$

A state observer is an auxiliary dynamical systems that mirrors the behavior of a physical system. One of the most known is the Luenberger observer which has the form

$$\dot{\hat{x}}(t) = Ax(t) + Bu(t) + H\left(y(t) - C\hat{x}(t)\right)$$

In order to guarantee the correct estimation of the state we need that the estimation error $e(t) = x(t) - \hat{x}(t)$ converges asymptotically to zero, this can be reached trough Lyapunov's stability with:

$$\dot{e} = (A - HC)e(t)$$

In this case $\lim_{t\to\infty} e(t) = 0$ if the matrix $(A - HC)$ is Hurwitz.

A functional observer estimates a function of the state vector without necessarily estimate the full state.

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + Fy(t) + Hu(t) \\ \hat{z}(t) &= P\zeta(t) + Qy(t) \end{aligned} \tag{2.34}$$

The estimation error $e(t) = \hat{z}(t) - z(t)$ converges asymptotically to zero if the matrix N in Hurwitz and there exists a matrix T such that the following conditions are satisfied:

$$\begin{aligned} NTE + FC - TA &= 0 \\ H - TB &= 0 \\ PTE + QC - L &= 0 \end{aligned}$$

Functional observers present the following advantages in comparison to full order state observers

- We can reduce the order of complexity of the observer.
- We can estimate a specific number of states.
- Since control signals are often linear combinations of the states, we can estimate a control signal for the stability of the system.

Finite-time convergence

For the case of continuous-time observers, the convergence of the estimation error to zero is always asymptotic with time. The convergence rate is exponential and can be modified by choosing the observer eigenvalues. In contrast, the observation problem in a discrete-time setting allows the choice of eigenvalues of zero and thereby a transient evolution which converges in a given number of steps, thus converging in a finite time. The guaranteed convergence time is then n times the sampling time and can be assigned by choosing the latter [Engel and Kreisselmeier, 2002]. Finite-convergence can be achieved in continuous time estimation by taking advantage of the discrete-time observers capabilities.

The following Lemma will be used later in this document.

Lemma 4 *[Zhang et al., 2016] For two given matrices M_1, $M_2 \in \mathbb{R}^{n\times n}$, if $Re\left(\lambda_i(M_2)\right) < \sigma < Re\left(\lambda_j(M_1)\right) < 0$, $i, j = 1, 2, ..., n$ for $\sigma < 0$, then matrix*

$$\begin{bmatrix} I & e^{M_1 h} \\ I & e^{M_2 h} \end{bmatrix}$$

is non singular for any $h > 0$

2.5 Conclusion

Chapter 2 has laid the cornerstone for the research journey undertaken in this thesis, meticulously examining the theoretical foundations crucial to understanding functional observers and control strategies for descriptor systems. By traversing the intricacies of descriptor systems in Section 2.1, we gained insights into their structures through illustrative examples and unveiled key properties that serve as the building blocks for subsequent analyses.

The exploration of Linear Matrix Inequality (LMI) regions in Section 2.2, with a specific focus on D-stability, has provided a mathematical framework for assessing system stability. This mathematical lens is instrumental in evaluating the robustness and reliability of control strategies, laying the groundwork for the subsequent chapters.

The extension to Takagi-Sugeno systems in Section 2.3 broadens the theoretical scope, demonstrating their applicability in addressing nonlinear complexities. The inclusion of functional observers in Section 2.4 completes the theoretical framework, highlighting their pivotal role in system observation and control. This comprehensive overview equips us with a versatile set of tools essential for advancing the understanding and application of functional observers and control strategies within the realm of descriptor systems.

As we move forward in the subsequent chapters, the theoretical foundation established here will serve as a guiding beacon, enabling us to delve into more specific aspects of functional observers and control for descriptor systems. By providing this theoretical groundwork, Chapter 2 paves the way for the practical implementation and refinement of these techniques, offering valuable insights for researchers and practitioners seeking to navigate the complexities of descriptor systems in diverse real-world scenarios.

Chapter 3

Functional observers for LTI systems

Contents

In practical applications, a myriad of control functions, such as fault diagnosis and monitoring, necessitate the manipulation of the state vector $x(t)$. However, obtaining the complete vector is often unfeasible due to exorbitant measurement costs or physical constraints. In such scenarios, an estimator proves invaluable in generating an estimate $\hat{x}(t)$ of the state vector $x(t)$. With the increasing complexity of controlled systems, there is a growing emphasis on ensuring safety and reliability in fault detection and estimation schemes.

Faults, defined as unpredictable alterations in the system's behavior leading to performance degradation, have become a focal point in control system design. The development of effective fault-tolerant control hinges on obtaining precise estimates of both the state variables and faults. Finite-time convergence has emerged as a captivating area of study [Engel and Kreisselmeier, 2002, Zhang et al., 2016], finding applications in diverse fields such as communication systems [Perruquetti et al., 2008], mechanical systems [Davila et al., 2005], and nonlinear systems [Shen and Xia, 2008]. The significance of finite-time convergence is particularly pronounced in fault estimation, given the prevalent need for rapid response constraints in practical control systems.

3.1 Functional observer for descriptor systems

In this section, we introduce a finite-time convergent estimator leveraging the framework of generalized functional observers. Capitalizing on the inherent structure of functional observers, we present a reduced-order observer designed to concurrently provide finite-time estimations of both the system's state and any faults present. This innovative approach enhances the efficiency and accuracy of state and fault estimation within a defined finite time period.

3.1.1 Class of systems considered

Let us consider the linear continuous time-invariant descriptor system described by the following equation:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ z(t) &= Lx(t) \end{aligned} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $u(t) \in \mathbb{R}^m$ is the input of the system, $y(t) \in \mathbb{R}^p$ is the measured output, and $z(t) \in \mathbb{R}^q$ is a function of the state to be estimated. Matrix $E \in \mathbb{R}^{n\times n}$ is singular, and A, B, C and L are known constant matrices of appropriate dimensions. Let $\mathsf{rank}(E) = r < n$ and $E^{\perp} \in \mathbb{R}^{r_1\times n}$ be a full row rank matrix such that $E^{\perp}E = 0$, where $r_1 = n - r$.

Lemma 5 *[Garcia and Bernussou, 1995] The eigenvalues of a given matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$, belong to the disk region $\mathcal{Q}(\alpha, \tau)$ with center α and radius τ if and only if, there exists a symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{n\times n}$ such that the following condition holds:*

$$\begin{bmatrix} -\mathcal{P}^{-1} & \frac{1}{\tau}(\mathcal{A} - \alpha I) \\ (*) & -\mathcal{P} \end{bmatrix} < 0 \tag{3.2}$$

By pre-multiplying and post-multiplying this inequality by $\begin{bmatrix} \mathcal{P} & 0 \\ 0 & I \end{bmatrix}$, we obtain the following equivalent LMI.

$$\begin{bmatrix} -\mathcal{P} & \frac{1}{\tau}\mathcal{P}(\mathcal{A} - \alpha I) \\ (*) & -\mathcal{P} \end{bmatrix} < 0 \tag{3.3}$$

By applying the result of Lemma 5 to two matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ we obtain the result of the following lemma.

Lemma 6 *[Engel and Kreisselmeier, 2002] For two matrices $\mathcal{A}_1$, $\mathcal{A}_2 \in \mathbb{R}^{n\times n}$, if $Re\,(\lambda_i(\mathcal{A}_2)) < \sigma < Re\,(\lambda_j(\mathcal{A}_1)) < 0$, $i, j = 1, 2, ..., n$ for $\sigma < 0$, then the matrix*

$$\begin{bmatrix} I & e^{\mathcal{A}_1 h} \\ I & e^{\mathcal{A}_2 h} \end{bmatrix} \tag{3.4}$$

is nonsingular for any $h > 0$, $\lambda_{i,j}(\mathcal{A}_{1,2})$ denotes the eigenvalues of matrix $\mathcal{A}_{1,2}$, respectively.

The following Lemma allows to parameterize all the solutions of an LMI.

Lemma 7 *[Skelton et al., 1997] Let matrices $\mathcal{B}$, $\mathcal{C}$, $\mathcal{Q} = \mathcal{Q}^T$ be given, then the following statements are equivalent:*

1. *There exists a matrix $\mathcal{X}$ satisfying*

$$\mathcal{B}\mathcal{X}\mathcal{C} + (\mathcal{B}\mathcal{X}\mathcal{C})^T + \mathcal{Q} < 0$$

2. *The following conditions hold*

$$\mathcal{B}^{\perp}\mathcal{Q}\mathcal{B}^{\perp T} < 0 \text{ or } \mathcal{B}\mathcal{B}^T > 0$$

$$\mathcal{C}^{T\perp}\mathcal{Q}\mathcal{C}^{T\perp T} < 0 \text{ or } \mathcal{C}^T\mathcal{C} > 0$$

where ^{T} denotes the transpose of the matrix. Suppose the above statements hold. Let r_b and r_c be the ranks of $\mathcal{B}$ and $\mathcal{C}$, respectively, and ($\mathcal{B}_l$,$\mathcal{B}_r$) and ($\mathcal{C}_l$,$\mathcal{C}_r$) be any full rank factors of $\mathcal{B}$ and $\mathcal{C}$, i.e., $\mathcal{B} = \mathcal{B}_l\mathcal{B}_r$, $\mathcal{C} = \mathcal{C}_l\mathcal{C}_r$. Then all matrices $\mathcal{X}$ in statement (1) of Lemma 7 are given by

$$\mathcal{X} = \mathcal{B}_r^+\mathcal{K}\mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_r^+\mathcal{B}_r\mathcal{Z}\mathcal{C}_l\mathcal{C}_l^+$$

where $\mathcal{Z}$ is an arbitrary matrix, the symbol $^+$ denotes the generalized inverse.

$$\mathcal{K} \triangleq -\mathcal{R}^{-1}\mathcal{B}_l^T\Theta\mathcal{C}_r^T(\mathcal{C}_r\Theta\mathcal{C}_r^T)^{-1} + \mathcal{S}^{1/2}\mathcal{L}(\mathcal{C}_r\Theta\mathcal{C}_r^T)^{-1/2}$$

$$\mathcal{S} \triangleq \mathcal{R}^{-1} - \mathcal{R}^{-1}\mathcal{B}_l^T[\Theta - \Theta\mathcal{C}_r^T(\mathcal{C}_r\Theta\mathcal{C}_r^T)^{-1}\mathcal{C}_r\Theta]\mathcal{B}_l\mathcal{R}^{-1}$$

where $\mathcal{L}$ is an arbitrary matrix such that $||\mathcal{L}|| < 1$ and $\mathcal{R}$ is an arbitrary positive definite matrix such that

$$\Theta \triangleq (\mathcal{B}_l\mathcal{R}^{-1}\mathcal{B}_l^T - \mathcal{Q})^{-1} > 0$$

Let us consider a class of descriptor linear time-invariant systems of the form (3.1). In the sequel we assume that system (3.1) is regular and the triplet (E, A, C) is partially impulse observable with respect to matrix L which is equivalent to the following assumption .

Assumption 1 *[Darouach, 2012] Functional observability allows us to verify if the linear function of the states can be estimated, from the information available from the input and output, this is satisfied if the following condition holds.*

$$rank \begin{bmatrix} L \\ E \\ E^{\perp}A \\ C \end{bmatrix} = rank \begin{bmatrix} E \\ E^{\perp}A \\ C \end{bmatrix}$$

The reconstruction of the function $z(t)$ in finite-time can be obtained by using two functional generalized dynamical observers of the form:

$$\begin{aligned} \dot{\zeta}_i(t) &= N_i\zeta_i(t) + J_i v_i(t) + F_i \begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} + H_i u(t) \\ \dot{v}_i(t) &= S_i\zeta_i(t) + G_i v_i(t) + M_i \begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \\ \hat{z}_i(t) &= P_i\zeta_i(t) + Q_i \begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \end{aligned} \tag{3.5}$$

for $i = 1, 2$, where $\zeta_i(t) \in \mathbb{R}^{q_0}$ are the states of the observers, $v_i(t) \in \mathbb{R}^{q_1}$ are auxiliary vectors and $\hat{z}_i(t) \in \mathbb{R}^{q}$ are the estimate of $z(t)$. N_i, J_i, F_i, H_i, S_i, G_i, M_i, P_i and Q_i are constant matrices of appropriate dimensions to be determined. Both observers have the same structure but different parameter matrices. Before continuing, we make the following remarks.

Remark 1 *For standard systems where $E = I$ Assumption 1 is always satisfied. Also when $L = I$, the case of the full-order observer is presented, and Assumption 1 is reduced to $rank \begin{bmatrix} E \\ E^{\perp}A \\ C \end{bmatrix} = n$ or $rank \begin{bmatrix} E & A \\ 0 & C \\ 0 & E \end{bmatrix} = n + rank(E)$, according to Lemma 3, which is the impulse observability of the triplet (E, A, C).*

Remark 2 *The form of the observer* (3.5) *is general and the Proportional Observer (PO) and the Proportional Integral observer (PIO) can be obtained as particular cases of (3.5). In fact.*

- *For $J_i = 0$, $S_i = 0$, $M_i = 0$ and $G_i = 0$, we obtain the P Observer.*
- *For $S_i = 0$ the observer becomes a PI Observer.*

The form of the observer given in (3.5) *presents an additional parameters which can introduce some slack variables, these variables can be used to realize other performances of the observer.*

3.1.2 Finite-Time Generalized Functional Observer Design

Consider parameter matrices $T_i \in \mathbb{R}^{q_0 \times n}$ and define the transformed error vectors

$$\varepsilon_i(t) = \zeta_i(t) - T_i E x(t) \tag{3.6}$$

Since $E^{\perp}E = 0$, it can be seen from (3.1) that, $E^{\perp}Ax(t) = -E^{\perp}Bu(t)$, such that the derivative of (3.6) is given by

$$\dot{\varepsilon}_i(t) = N_i\varepsilon_i(t) + \left(N_iT_iE + F_i\begin{bmatrix}E^{\perp}A\\ C\end{bmatrix} - T_iA\right)x(t) + (H_i - T_iB)u(t) + J_iv_i(t) \tag{3.7}$$

from (3.5), $\dot{v}_i(t)$ can be rewritten as

$$\dot{v}_i(t) = S_i\varepsilon_i(t) + G_iv_i(t) + \left(M_i\begin{bmatrix}E^{\perp}A\\ C\end{bmatrix} + S_iT_iE\right)x(t) \tag{3.8}$$

If the following conditions are satisfied

a. $N_iT_iE + F_i\begin{bmatrix}E^{\perp}A\\ C\end{bmatrix} - T_iA = 0$

b. $H_i = T_iB$

c. $S_iT_iE + M_i\begin{bmatrix}E^{\perp}A\\ C\end{bmatrix} = 0, \quad \text{for} \quad \text{i=1,2}$

d. $P_iT_iE + Q_i\begin{bmatrix}E^{\perp}A\\ C\end{bmatrix} = L, \quad \text{for} \quad \text{i=1,2}$

then (3.7) and (3.8) can be written as

$$\dot{\varepsilon}_i(t) = N_i\varepsilon_i(t) + J_iv_i(t) \tag{3.9}$$
$$\dot{v}_i(t) = S_i\varepsilon_i(t) + G_iv_i(t) \tag{3.10}$$

By defining an augmented state vector $\vartheta_i(t) = \begin{bmatrix}\varepsilon_i(t)\\ v_i(t)\end{bmatrix}$, (3.9) and (3.10) can be rewritten as:

$$\dot{\vartheta}_i(t) = \underbrace{\begin{bmatrix}N_i & J_i\\ S_i & G_i\end{bmatrix}}_{\mathbb{A}_i}\vartheta_i(t) \tag{3.11}$$

Now, from (3.5) we obtain

$$\hat{z}_i(t) = P_i\varepsilon_i(t) + \left(P_iT_iE + Q_i\begin{bmatrix}E^{\perp}A\\ C\end{bmatrix}\right)x(t) \tag{3.12}$$

If condition d. is satisfied, then

$$e_i(t) = P_i\varepsilon_i(t) \tag{3.13}$$

where $e_i(t) = \hat{z}_i(t) - z(t)$.
It can be seen that, if matrices $\mathbb{A}_i$ from (3.11) are Hurwitz, then $\lim_{t\to\infty}\vartheta_i(t) = 0$ and $\lim_{t\to\infty}e_i(t) = 0$, for $i = 1, 2$.

Observer parameterization

From conditions **a.** to **d.**, it can be seen that the design problem of the two observers is reduced to find the matrices N_i, J_i, F_i, H_i, S_i, G_i, M_i, P_i, Q_i and T_i such that $\mathbb{A}_i$ is Hurwitz.

Define matrix $\Gamma = \begin{bmatrix} E \\ E^{\perp}A \\ C \end{bmatrix}$ and let $R_i \in \mathbb{R}^{q_0 \times n}$, for $i = 1, 2$, be a full row rank matrix such that $\mathsf{rank} \begin{bmatrix} R_i \\ \Gamma \end{bmatrix} = \mathsf{rank}(\Gamma)$. Also let us define the matrix $\Sigma_i = \begin{bmatrix} R_i \\ E^{\perp}A \\ C \end{bmatrix}$. In this case there always exists two matrices T_i and K_i such that

$$T_i E + K_i \begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} = R_i \tag{3.14}$$

which can be rewritten as

$$\begin{bmatrix} T_i & K_i \end{bmatrix} \Gamma = R_i \tag{3.15}$$

the general solution of (3.15) is given by

$$\begin{bmatrix} T_i & K_i \end{bmatrix} = R_i \Gamma^+ - Z_i \left(I - \Gamma\Gamma^+ \right) \tag{3.16}$$

where Γ^+ is the generalized inverse of Γ.
Equation (3.16) leads to

$$T_i = T_{i1} - Z_i T_2 \tag{3.17}$$

$$K_i = K_{i1} - Z_i K_2 \tag{3.18}$$

where Z_i is a matrix of appropriate dimensions with arbitrary elements, and $T_{i1} = R_i \Gamma^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$, $T_2 = (I - \Gamma\Gamma^+) \begin{bmatrix} I \\ 0 \end{bmatrix}$, $K_{i1} = R_i \Gamma^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $K_2 = (I - \Gamma\Gamma^+) \begin{bmatrix} 0 \\ I \end{bmatrix}$.
From condition **a.** and (3.14), we have:

$$N_i \left(R_i - K_i \begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} \right) + F_i \begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} = T_i A \tag{3.19}$$

which can be written as

$$\begin{bmatrix} N_i & \tilde{K}_i \end{bmatrix} \Sigma_i = T_i A \tag{3.20}$$

where $\tilde{K}_i = F_i - N_i K_i$. If $\mathsf{rank} \begin{bmatrix} \Sigma_i \\ T_i A \end{bmatrix} = \mathsf{rank}(\Sigma_i)$, the general solution to (3.20) is

$$\begin{bmatrix} N_i & \tilde{K}_i \end{bmatrix} = T_i A \Sigma_i^+ - Y_{i1}(I - \Sigma_i \Sigma_i^+) \tag{3.21}$$

where Y_{i1} is a matrix of appropriate dimensions with arbitrary elements.
By replacing (3.17) into (3.21), we obtain

$$N_i = N_{i1} - Z_i N_{i2} - Y_{i1} N_{i3} \tag{3.22}$$

$$\tilde{K}_i = \tilde{K}_{i1} - Z_i \tilde{K}_{i2} - Y_{i1} \tilde{K}_{i3} \tag{3.23}$$

where $N_{i1} = T_{i1}A\Sigma_i^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$, $N_{i2} = T_{i2}A\Sigma_i \begin{bmatrix} I \\ 0 \end{bmatrix}$, $N_{i3} = (I - \Sigma_i\Sigma_i^+) \begin{bmatrix} I \\ 0 \end{bmatrix}$, $\tilde{K}_{i1} = T_{i1}A\Sigma_i^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$, $\tilde{K}_{i2} = T_{i2}A\Sigma_i^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $\tilde{K}_{i3} = (I - \Sigma_i\Sigma_i^+) \begin{bmatrix} 0 \\ I \end{bmatrix}$.
Since $F_i = \tilde{K}_i + N_iK_i$, we have

$$\begin{aligned} F_i &= \tilde{K}_{i1} + N_{i1}K_i - Z_i(\tilde{K}_{i2} - N_{i2}K_i) - Y_{i1}(\tilde{K}_3 - N_{i3}K_i) \\ F_i &= F_{i1} - Z_iF_{i2} - Y_{i1}F_{i3} \end{aligned} \tag{3.24}$$

where $F_{i1} = T_{i1}A\Sigma_i^+ \begin{bmatrix} K_i \\ I \end{bmatrix}$, $F_{i2} = T_{i2}A\Sigma_i^+ \begin{bmatrix} K_i \\ I \end{bmatrix}$ and $F_{i3} = (I - \Sigma_i\Sigma_i^+) \begin{bmatrix} K_i \\ I \end{bmatrix}$.
From (3.14) we have

$$\begin{bmatrix} T_iE \\ E^\perp A \\ C \end{bmatrix} = \begin{bmatrix} I & -K_i \\ 0 & I \end{bmatrix} \Sigma_i \tag{3.25}$$

Conditions **c.** and **d.** can be written as

$$\begin{bmatrix} S_i & M_i \\ P_i & Q_i \end{bmatrix} \begin{bmatrix} T_iE \\ E^\perp A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ L \end{bmatrix} \tag{3.26}$$

If we substitute (3.25) into (3.26), we obtain

$$\begin{bmatrix} S_i & M_i \\ P_i & Q_i \end{bmatrix} \begin{bmatrix} I & -K_i \\ 0 & I \end{bmatrix} \Sigma_i = \begin{bmatrix} 0 \\ L \end{bmatrix} \tag{3.27}$$

By using the fact that $\begin{bmatrix} I & -K_i \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & K_i \\ 0 & I \end{bmatrix}$ and considering that $rank \begin{bmatrix} \Sigma_i \\ L \end{bmatrix} = rank(\Sigma_i)$, the general solution to (3.27) is given by

$$\begin{bmatrix} S_i & M_i \\ P_i & Q_i \end{bmatrix} = \left(\begin{bmatrix} 0 \\ L \end{bmatrix} \Sigma_i^+ - Y_i \left(I - \Sigma_i\Sigma_i^+\right) \right) \begin{bmatrix} I & K_i \\ 0 & I \end{bmatrix} \tag{3.28}$$

where $Y_i = \begin{bmatrix} Y_{i2} \\ Y_{i3} \end{bmatrix}$ is a matrix of appropriate dimensions with arbitrary elements. In this case, the particular solutions for S_i, M_i, P_i and Q_i are given by

$$S_i = -Y_{i2}N_{i3} \tag{3.29}$$

$$M_i = -Y_{i2}F_{i3} \tag{3.30}$$

$$P_i = P_{i1} - Y_{i3}N_{i3} \tag{3.31}$$

$$Q_i = Q_{i1} - Y_{i3}F_{i3} \tag{3.32}$$

where $P_{i1} = L\Sigma_i^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$ and $Q_{i1} = L\Sigma_i^+ \begin{bmatrix} K_i \\ I \end{bmatrix}$.
We can see that the design is independent of the parameter matrices Y_{i3}, which can take an

arbitrarily value. For simplicity we can take $Y_{i3} = 0$.
Now, by using (3.22) and (3.29), the error dynamics (3.11) can be rewritten as

$$\dot{\vartheta}_i(t) = (\mathbb{A}_{i1} - \mathbb{Y}_i \mathbb{A}_{i2})\vartheta_i(t) \tag{3.33}$$

where $\mathbb{A}_{i1} = \begin{bmatrix} N_{i1} - Z_i N_{i2} & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{Y} = \begin{bmatrix} Y_{i1} & J_i \\ Y_{i2} & G_i \end{bmatrix}$ and $\mathbb{A}_{i2} = \begin{bmatrix} N_{i3} & 0 \\ 0 & -I \end{bmatrix}$.
The problem is now reduced to find matrices $\mathbb{Y}_i$ and Z_i such that matrices $\mathbb{A}_i$ are Hurwitz. This can be reached by using any pole placement method or a linear matrix inequality (LMI) approach.
Now, the necessary and sufficient condition for the existence of matrices $\mathbb{Y}_i$ such that matrices $\mathbb{A}_i$ are Hurwitz is that the pair $(\mathbb{A}_{i2}, \mathbb{A}_{i1})$ must be detectable.

The following Lemma gives the necessary and sufficient conditions for the detectability of the pair $(\mathbb{A}_{i2}, \mathbb{A}_{i1})$.

Lemma 8 *The following statements are equivalent*

1. *The pair* $(\mathbb{A}_{i2}, \mathbb{A}_{i1})$ *is detectable.*
2. $\mathsf{rank} \begin{bmatrix} \lambda I - N_{i1} + Z_i N_{i2} \\ N_{i3} \end{bmatrix} = q_0, \forall \lambda \in \mathbb{C}, \mathsf{Re}(\lambda) \geq 0$

Proof 2 *The pair* $(\mathbb{A}_{i1}, \mathbb{A}_{i2})$ *is detectable if and only if*

$$\mathsf{rank} \begin{bmatrix} \lambda I - \mathbb{A}_{i1} \\ \mathbb{A}_{i2} \end{bmatrix} = q_0 + q_1, \forall \lambda \in \mathbb{C}, \mathsf{Re}(\lambda) \geq 0$$

or equivalently

$$\mathsf{rank} \begin{bmatrix} \lambda I - N_{i1} + Z_i N_{i2} & 0 \\ 0 & \lambda I \\ N_{i3} & 0 \\ 0 & -I \end{bmatrix} = q_0 + q_1, \forall \lambda \in \mathbb{C}, \mathsf{Re}(\lambda) \geq 0 \tag{3.34}$$

Remark 3 *From the result of the above lemma it is easy to show that a sufficient condition for matrix* $\mathbb{A}_i$ *to be Hurwitz is*

$$\mathsf{rank} \begin{bmatrix} \lambda I - N_{i1} \\ N_{i2} \\ N_{i3} \end{bmatrix} = q_0, \forall \lambda \in \mathbb{C}, \mathsf{Re}(\lambda) \geq 0. \tag{3.35}$$

Finite-time observer design

From the precedent results, the design of a generalized finite time observer is reduced to determine the parameters $\mathbb{Y}_i$ and Z_i such that the eigenvalues of matrices $\mathbb{A}_i$ are inside two disks regions $\mathcal{Q}_1(\alpha_1, \tau_1)$ and $\mathcal{Q}_2(\alpha_2, \tau_2)$ by using Lemma 5. The following theorem gives the sufficient conditions in LMIs formulation to realize this objective.

Theorem 1 *Under assumption 1 and condition* (3.35)*, let* $\mathcal{Q}_i(\alpha_i, \tau_i), i = 1, 2$ *be two disks region, then there exist parameters matrices* $\mathbb{Y}_i$ *and* Z_i *such that the eigenvalues of matrices* $\mathbb{A}_i$ *are inside these regions, if there exists two symmetric positive definite matrices* $X_i = \begin{bmatrix} X_{i1} & X_{i1} \\ X_{i1} & X_{i2} \end{bmatrix} \in \mathbb{R}^{(q_0+q_1)\times(q_0+q_1)}$ *such that the following LMIs are satisfied:*

$$\begin{bmatrix} -X_i & \tilde{\mathbb{A}}_i \\ (*) & -\mathbb{A}_{i2}^{T\perp} X_i \mathbb{A}_{i2}^{T\perp T} \end{bmatrix} < 0 \tag{3.36}$$

where $\tilde{\mathbb{A}}_i = \frac{1}{\tau_i} \begin{bmatrix} (X_{i1}N_{i1} - W_i N_{i2} - \alpha_i X_{i1}) N_{i3}^{T\perp T} \\ (X_{i1}N_{i1} - W_i N_{i2} - \alpha_i X_{i1} i) N_{i3}^{T\perp T} \end{bmatrix}$, $W_i = X_{i1} Z_i$ *and matrices* $\mathbb{Y}_i$ *can be parameterized as it follows:*

$$\mathbb{Y}_i = -\tau_i \left(\mathcal{B}_{ir}^{+} \mathcal{K}_i \mathcal{C}_{il}^{+} + \mathcal{Z}_i - \mathcal{B}_{ir}^{+} \mathcal{B}_{ir} \mathcal{Z}_i \mathcal{C}_{il} \mathcal{C}_{il}^{+} \right) \tag{3.37}$$

where

$$\mathcal{K}_i = -\mathcal{R}_i^{-1} \mathcal{B}_{il}^T \Theta_i \mathcal{C}_{ir}^T (\mathcal{C}_{ir} \Theta_i \mathcal{C}_{ir}^T)^{-1} + \mathcal{S}_i^{1/2} \mathcal{L}_i (\mathcal{C}_{ir} \Theta_i \mathcal{C}_{ir}^T)^{-1/2} \tag{3.38}$$

$$\mathcal{S}_i = \mathcal{R}_i^{-1} - \mathcal{R}_i^{-1} \mathcal{B}_{il}^T [\Theta_i - \Theta_i \mathcal{C}_{ir}^T (\mathcal{C}_{ir} \Theta_i \mathcal{C}_{ir}^T)^{-1} \mathcal{C}_{ir} \Theta_i] \mathcal{B}_{il} \mathcal{R}_i^{-1} \tag{3.39}$$

and where $\mathcal{L}_i$ *is an arbitrary matrix such that* $||\mathcal{L}_i|| < 1$ *and* $\mathcal{R}_i$ *is an arbitrary positive definite matrix such that*

$$\Theta_i = (\mathcal{B}_{il} \mathcal{R}_i^{-1} \mathcal{B}_{il}^T - \mathcal{Q}_i)^{-1} > 0$$

for $i = 1, 2$.

Proof 3 *Under Assumption 1 and Condition* (3.35)*, from Lemma 5 the eigenvalues of matrices* $\mathbb{A}_i$ *are inside the two disks region* $\mathcal{Q}_i(\alpha_i, \tau_i), i = 1, 2$*, if the following LMIs are satisfied*

$$\begin{bmatrix} -X_i & \frac{1}{\tau_i}(X_i \mathbb{A}_i - \alpha_i X_i) \\ (*) & -X_i \end{bmatrix} < 0 \tag{3.40}$$

where the symbol $(*)$ *denotes the transpose elements in the symmetric positions. Also* (3.40) *can be written as*

$$\mathcal{B}_i \mathcal{X}_i \mathcal{C}_i + (\mathcal{B}_i \mathcal{X}_i \mathcal{C}_i)^T + \mathcal{Q}_i < 0 \tag{3.41}$$

where $\mathcal{B}_i = \begin{bmatrix} X_i \\ 0 \end{bmatrix}$, $\mathcal{X}_i = -\frac{1}{\tau_i} \mathbb{Y}_i$, $\mathcal{C}_i = \begin{bmatrix} 0 & \mathbb{A}_{i2} \end{bmatrix}$, *and* $\mathcal{Q}_i = \begin{bmatrix} -X_i & \frac{1}{\tau_i}(X_i \mathbb{A}_{i1} - \alpha_i X_i) \\ (*) & -X_i \end{bmatrix}$. *From Lemma 7, the solvability conditions of* (3.41) *are*

$$\mathcal{B}_i^{\perp} \mathcal{Q}_i \mathcal{B}_i^{\perp T} < 0 \tag{3.42}$$

$$\mathcal{C}_i^{T\perp} \mathcal{Q}_i \mathcal{C}_i^{T\perp T} < 0 \tag{3.43}$$

since $\mathcal{B}_i^{\perp} = \begin{bmatrix} 0 & I \end{bmatrix}$, *inequality* (3.90) *becomes* $X_i > 0$ *and considering* $\mathcal{C}_i^{T\perp} = \begin{bmatrix} I & 0 \\ 0 & \mathbb{A}_{i2}^{T\perp} \end{bmatrix}$ *we obtain*

$$\begin{bmatrix} -X_i & \frac{1}{\tau_i}(X_i \mathbb{A}_{i1} - \alpha_i X_i) \mathbb{A}_{i2}^{T\perp T} \\ (*) & -\mathbb{A}_{i2}^{T\perp} X_i \mathbb{A}_{i2}^{T\perp T} \end{bmatrix} < 0 \tag{3.44}$$

By developing the expression $X_i\mathbb{A}_{i1} - \alpha_i X_i$, we obtain

$$X_i\mathbb{A}_{i1} - \alpha_i X_i = \begin{bmatrix} X_{i1}N_{i1} - W_iN_{i2} - \alpha_i X_{i1} & -\alpha_i X_{i1} \\ X_{i1}N_{i1} - W_iN_{i2} - \alpha_i X_{i1} & -\alpha_i X_{i2} \end{bmatrix} \tag{3.45}$$

where $W_i = X_{i1}Z_i$. Since, $\mathbb{A}_{i2}^{T\perp} = \begin{bmatrix} N_{i3}^{T\perp} & 0 \end{bmatrix}$, then $(X_i\mathbb{A}_{i1} - \alpha_i X_i)\mathbb{A}_{i2}^{T\perp T}$ can be written as

$$(X_i\mathbb{A}_{i1} - \alpha_i X_i)\mathbb{A}_{i2}^{T\perp T} = \begin{bmatrix} (X_{i1}N_{i1} - W_iN_{i2} - \alpha_i X_{i1})N_{i3}^{T\perp T} \\ (X_{i1}N_{i1} - W_iN_{i2} - \alpha_i X_{i1})N_{i3}^{T\perp T} \end{bmatrix}$$

Let $\tilde{\mathbb{A}}_i = \frac{1}{\tau_i}\begin{bmatrix} (X_{i1}N_{i1} - W_iN_{i2} - \alpha_i X_{i1})N_{i3}^{T\perp T} \\ (X_{i1}N_{i1} - W_iN_{i2} - \alpha_i X_{i1})N_{i3}^{T\perp T} \end{bmatrix}$, then (3.44) can be written as

$$\begin{bmatrix} -X_i & \tilde{\mathbb{A}}_i \\ \tilde{\mathbb{A}}_i^T & -\mathbb{A}_{i2}^{T\perp} X_i \mathbb{A}_{i2}^{T\perp T} \end{bmatrix} < 0 \tag{3.46}$$

By using Lemma 7 we obtain the parameterization of matrix $\mathbb{Y}_i$ as in (3.37) which completes the proof.

Finite-Time estimator

Consider the estimation error $e_i(t) = \hat{z}_i(t) - z(t)$ which can be written as $e_i(t) = P_i\varepsilon_i(t)$. By using Eq. (3.11), the estimation error becomes

$$e_i(t) = \bar{P}_i\vartheta_i(t), \, for \, i = 1, 2 \tag{3.47}$$

where $\bar{P}_i = \begin{bmatrix} P_i & 0 \end{bmatrix}$.

We can now express the following equations

$$\begin{aligned} \hat{z}_1(t) &= \bar{P}_1\vartheta_1(t) + z(t) \\ \hat{z}_2(t) &= \bar{P}_2\vartheta_2(t) + z(t) \\ \hat{z}_1(t-h) &= \bar{P}_1\vartheta_1(t-h) + z(t-h) \\ \hat{z}_2(t-h) &= \bar{P}_2\vartheta_2(t-h) + z(t-h) \end{aligned} \tag{3.48}$$

By using (3.11), we have

$$\vartheta_i(t) = e^{\mathbb{A}_i(t-t_0)}\vartheta_i(t_0) \tag{3.49}$$

for $t_0 = t - h$, with h being a scalar positive we obtain

$$\vartheta_i(t) = e^{\mathbb{A}_i h}\vartheta_i(t-h) \tag{3.50}$$

By replacing (3.50) into (3.48), we obtain

$$\begin{aligned} \hat{z}_1(t) &= \bar{P}_1\vartheta_1(t) + z(t) \\ \hat{z}_2(t) &= \bar{P}_2\vartheta_2(t) + z(t) \\ \hat{z}_1(t-h) &= \bar{P}_1 e^{-\mathbb{A}_1 h}\vartheta_1(t) + z(t-h) \\ \hat{z}_2(t-h) &= \bar{P}_2 e^{-\mathbb{A}_2 h}\vartheta_2(t) + z(t-h) \end{aligned} \tag{3.51}$$

Now, consider the vectors $\hat{z}(t) = \begin{bmatrix}\hat{z}_1(t) & \hat{z}_2(t)\end{bmatrix}^T$, and $\vartheta(t) = \begin{bmatrix}\vartheta_1(t) & \vartheta_2(t)\end{bmatrix}^T$, and matrices $\bar{P} = \begin{bmatrix}\bar{P}_1 & 0 \\ 0 & \bar{P}_2\end{bmatrix}$, $e^{\mathbb{A}h} = \begin{bmatrix}e^{\mathbb{A}_1 h} & 0 \\ 0 & e^{\mathbb{A}_2 h}\end{bmatrix}$, and $\Psi = \begin{bmatrix}I \\ I\end{bmatrix}$, and replace them in (3.51)

$$\hat{z}(t) = \bar{P}\vartheta(t) + \Psi z(t) \tag{3.52}$$
$$\hat{z}(t-h) = \bar{P}e^{-\mathbb{A}h}\vartheta(t) + \Psi z(t-h) \tag{3.53}$$

by replacing (3.52) in (3.51), we obtain

$$\hat{z}(t-h) = \bar{P}e^{-\mathbb{A}h}\bar{P}^{+}\left(\hat{z}(t) - \Psi z(t)\right) + \Psi z(t-h) \tag{3.54}$$

which can be rewritten as

$$\hat{z}(t-h) - \Psi z(t-h) = \bar{Q}\left(\hat{z}(t) - \Psi z(t)\right) \tag{3.55}$$

where $\bar{Q} = \bar{P}e^{-\mathbb{A}h}\bar{P}^{+}$, and can be expressed as

$$\begin{bmatrix}\bar{Q}\Psi & \Psi\end{bmatrix}\begin{bmatrix}z(t) \\ -\hat{z}(t-h)\end{bmatrix} = \bar{Q}\hat{z}(t) - \hat{z}(t-h) \tag{3.56}$$

Then, $z(t)$ can be obtained by

$$\hat{z}(t) = \begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}\bar{Q}\Psi & \Psi\end{bmatrix}^{-1}\left(\bar{Q}\hat{z}(t) - \hat{z}(t-h)\right) \tag{3.57}$$

The existence of a finite-time convergence observer can be determined by matrix $\begin{bmatrix}\bar{Q}\Psi & \Psi\end{bmatrix}^{-1}$, in this case, this matrix exists if

$$\begin{aligned}\det\begin{bmatrix}\bar{Q}\Psi & \Psi\end{bmatrix} &\neq 0 \\ \det\begin{bmatrix}\bar{Q}_1 & I \\ \bar{Q}_2 & I\end{bmatrix} &\neq 0 \\ \det(\bar{Q}_1 - \bar{Q}_2) &\neq 0\end{aligned} \tag{3.58}$$

Remark 4 *From* (3.60) *, in the case of* $P_1 = P_2$*, the necessary condition is reduced to* $\lambda(\mathbb{A}_1) < \rho < \lambda(\mathbb{A}_2)$

Numerical example

The following example is given to illustrate the Functional Finite-Time observer performance and its capability to estimate a control law to stabilize a Descriptor Linear Time-Invariant system. Let us consider the following Descriptor system of the form (3.1) with:

$$E = \begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix} \quad A = \begin{bmatrix}-1 & 3 & 0 & 1 \\ 1 & -7 & 2.5 & 3 \\ 3 & -1 & 1 & 0 \\ 2 & -3.1 & -2.2 & 0.5\end{bmatrix}, B = \begin{bmatrix}1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0\end{bmatrix}, \text{ and } C = \begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}$$

as we can see, $\lambda(E,A) = \begin{bmatrix} 3.06 \\ -4.03-6.0\text{i} \\ -4.03+6.0\text{i} \end{bmatrix}$, this makes the system unstable. The Finite-time Functional observer is designed to stabilize the system. Matrix L is obtained by a simple pole placement method. In this case we choose $L = \begin{bmatrix} -30.09 & -8.39 & 67.22 & -1.96 \\ 12.83 & -6.5 & -47.24 & 24.11 \end{bmatrix}$ then the poles of the system are $\lambda(E, A-BL) = \begin{bmatrix} -5.02 \\ -10.84 \\ -274.643 \end{bmatrix}$. The performance of the Observer is tested in Simulation 1.

Simulation 1 ***Finite-time control law estimation***
The objective of this simulation is to estimate a control law for an unstable system of the form (3.1)*. The estimated control law is feeded to the system (see Figure 3.1). In this case, the role of the controller is satisfied by the functional observer, and the reference is set to* 0*.*

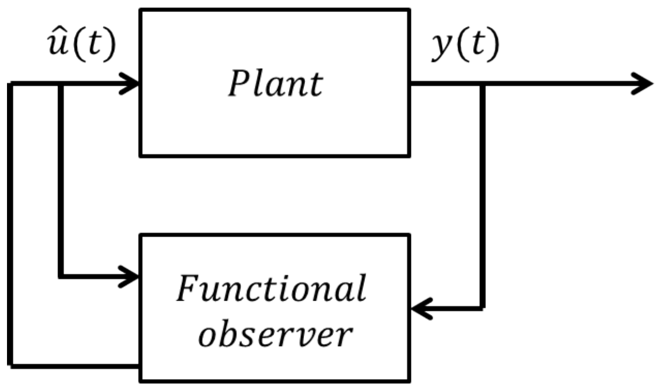


Figure 3.1: Block diagram of the control system

Settings*. The disks regions $\mathcal{Q}(\alpha_i, \tau_i)$ are chosen as $\alpha_1 = -5$, $\alpha_2 = -2$, $\tau_1 = \tau_2 = 0.5$, with a convergence time of* $0.5s$ *and initial condition $x(0) = \begin{bmatrix} -0.5 & 1 & 2.5 & -1 \end{bmatrix}^T$. The measurable outputs are $x_1(t)$, $x_2(t)$ and $x_4(t)$. The results are shown in Figure 3.2 and the control law of system is shown in Figure 3.3*

It can be seen in Figures 3.2 and 3.3 that the observer is capable to estimate a control law that stabilizes the system.

3.1.3 GDO design for LTI systems

The main contribution of the work is to propose a generalized asymptotic functional observer which extends the results presented in [Darouach, 2012]. Then a combination of two generalized functional observers is used to design a generalized finite-time functional observer. The obtained results are then applied to estimate a linear function of the state in a finite-time.

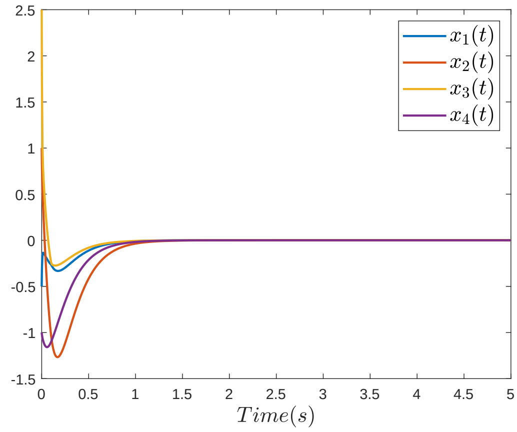


Figure 3.2: States of the LTI descriptor system

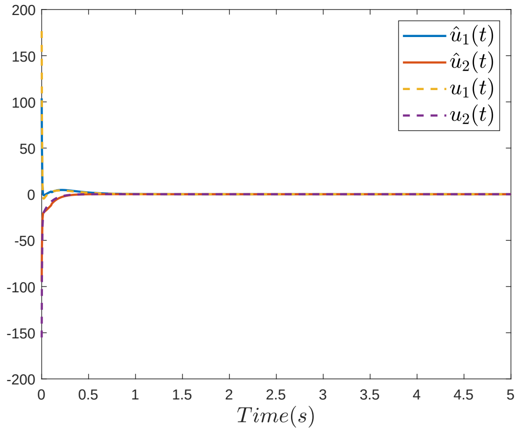


Figure 3.3: Estimated control law for the LTI descriptor system

So, a finite-time convergent estimator based on the generalized functional observer is given. We take the advantage of the generalized functional observer structure, to propose an observer capable to give a determined finite-time convergence functional estimation. A numerical example is given to illustrate our approach.

Preliminaries

In this section we present the preliminaries that are used in the sequel of this work.

Let us consider the linear continuous time-invariant system described by the following equation:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ z(t) &= Lx(t) \end{aligned} \tag{3.59}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $u(t) \in \mathbb{R}^m$ is the input of the system, $y(t) \in \mathbb{R}^p$ is the measured output, and $z(t) \in \mathbb{R}^q$ is a function of the state to be estimated. Matrices A, B, C and L are known constant matrices of appropriate dimensions. Also, in the sequel of the work, P^+denotes the generalized inverse of matrix P satisfying $PP^+P = P$. The symbol $P^\perp$ denotes a maximal row rank matrix such that $P^\perp P = 0$. When P is a full row rank matrix, $P^\perp = 0$.

Remark 5 *This is the case of Standard LTI systems, where $E = I$*

Now, let us consider a class of Linear Time-Invariant (LTI) systems of the form (3.59). The reconstruction of the function $z(t)$ in finite-time can be obtained by using two functional generalized dynamical observers of the form:

$$\begin{aligned} \dot{\zeta}_i(t) &= N_i\zeta_i(t) + J_i v_i(t) + F_i y(t) + H_i u(t) \\ \dot{v}_i(t) &= S_i\zeta_i(t) + G_i v_i(t) + M_i y(t) \\ \hat{z}_i(t) &= P_i\zeta_i(t) + Q_i y(t) \end{aligned} \tag{3.60}$$

for $i = 1, 2$, where $\zeta_i(t) \in \mathbb{R}^{q_0}$ are the states of the observers, $v_i(t) \in \mathbb{R}^{q_1}$ are auxiliary vectors and $\hat{z}_i(t) \in \mathbb{R}^q$ are the estimates of $z(t)$. N_i, J_i, F_i, H_i, S_i, G_i, M_i, P_i and Q_i are constant matrices of appropriate dimensions to be determined. Both observers have the same structure but different parameter matrices. Before continuing, we make the following remark.

Finite-Time Generalized Functional Observer Design

Consider parameter matrices $T_i \in \mathbb{R}^{q_0 \times n}$ and define the transformed error vectors

$$\varepsilon_i(t) = \zeta_i(t) - T_i x(t) \tag{3.61}$$

The derivative of $\varepsilon_i(t)$ is given by

$$\dot{\varepsilon}_i(t) = N_i\varepsilon_i(t) + (N_iT_i + F_iC - T_iA)\, x(t) + (H_i - T_iB)u(t) + J_i v_i(t) \tag{3.62}$$

$\dot{v}_i(t)$ in (3.60) can be rewritten as

$$\dot{v}_i(t) = S_i\varepsilon_i(t) + G_i v_i(t) + (M_iC + S_iT_i)\, x(t) \tag{3.63}$$

So, if the following conditions are satisfied

a. $N_iT_i + F_iC - T_iA = 0$

b. $H_i = T_i B$

c. $S_i T_i + M_i C = 0, \quad \text{for} \quad \text{i=1,2}$

then (3.62) and (3.63) can be written as

$$\dot{\varepsilon}_i(t) = N_i \varepsilon_i(t) + J_i v_i(t) \tag{3.64}$$
$$\dot{v}_i(t) = S_i \varepsilon_i(t) + G_i v_i(t) \tag{3.65}$$

By defining an augmented state vector $\vartheta_i(t) = \begin{bmatrix} \varepsilon_i(t) \\ v_i(t) \end{bmatrix}$, (3.64) and (3.65) can be rewritten as:

$$\dot{\vartheta}_i(t) = \underbrace{\begin{bmatrix} N_i & J_i \\ S_i & G_i \end{bmatrix}}_{\mathbb{A}_i} \vartheta_i(t) \tag{3.66}$$

Now, from (3.60) we obtain

$$\hat{z}_i(t) = P_i \varepsilon_i(t) + (P_i T_i + Q_i C)\, x(t) \tag{3.67}$$

considering that condition

d. $P_i T_i + Q_i C = L, \quad \text{for} \quad \text{i=1,2}$

is satisfied, then

$$e_i(t) = P_i \varepsilon_i(t) \tag{3.68}$$

where $e_i(t) = \hat{z}_i(t) - z(t)$.
It can be seen that, if matrix $\mathbb{A}_i$ from (3.66) is Hurwitz, then $\lim_{t\to\infty} \vartheta_i(t) = 0$ and $\lim_{t\to\infty} e_i(t) = 0$, for $i = 1, 2$.

Observer parameterization

From conditions **a.-d.**, it can be seen that the design problem of the two observers is reduced to find the matrices N_i, J_i, F_i, H_i, S_i, G_i, M_i, P_i, Q_i and T_i such that $\mathbb{A}_i$ is Hurwitz.
Define matrix $\Sigma = \begin{bmatrix} R \\ C \end{bmatrix}$. In this case there always exists two matrices T_i and K_i such that

$$T_i + K_i C = R \tag{3.69}$$

which can be rewritten as

$$T_i = R - K_i C \tag{3.70}$$

From condition **a.** and (3.69), we have:

$$\begin{aligned} N_i (R - K_i C) + F_i C - (R - K_i C) A &= 0 \\ N_i R + (F_i - N_i K_i) C + K_i C A &= RA \end{aligned} \tag{3.71}$$

let $K_{i1} = F_i - N_iK_i$, then (3.71) can be written as

$$\begin{bmatrix} N_i & K_{i1} \end{bmatrix} \Sigma = RA - K_iCA \tag{3.72}$$

the general solution to (3.72) is

$$\begin{bmatrix} N_i & K_{i1} \end{bmatrix} = (RA - K_iCA)\Sigma^+ - Y_{i1}(I - \Sigma\Sigma^+) \tag{3.73}$$

where Y_{i1} is a matrix of appropriate dimensions with arbitrary elements. Now, N_i and K_{i1} can be obtained as it follows

$$N_i = N_{11} - K_iN_{22} - Y_{i1}N_3 \tag{3.74}$$

$$K_{i1} = \tilde{K}_1 - K_i\tilde{K}_2 - Y_{i1}\tilde{K}_3 \tag{3.75}$$

where $N_{11} = RA\Sigma^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$, $N_2 = CA\Sigma^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$, $N_3 = (I - \Sigma\Sigma^+) \begin{bmatrix} I \\ 0 \end{bmatrix}$, $\tilde{K}_1 = RA\Sigma^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$, $\tilde{K}_2 = CA\Sigma^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $\tilde{K}_3 = (I - \Sigma\Sigma^+) \begin{bmatrix} 0 \\ I \end{bmatrix}$.

If we substitute (3.70) into conditions **c.** and **d.** we obtain

$$\begin{bmatrix} S_i(R - K_iC) + M_iC \\ P_i(R - K_iC) + Q_iC \end{bmatrix} = \begin{bmatrix} 0 \\ L \end{bmatrix} \tag{3.76}$$

Let $K_{i2} = (M_i - S_iK_i)$ and $K_{i3} = (Q_i - P_iK_i)$, then (3.76) can be written as

$$\begin{bmatrix} S_i & K_{i2} \\ P_i & K_{i3} \end{bmatrix} \Sigma = \begin{bmatrix} 0 \\ L \end{bmatrix} \tag{3.77}$$

the general solution to (3.130) is given by

$$\begin{bmatrix} S_i & K_{i2} \\ P_i & K_{i3} \end{bmatrix} = \begin{bmatrix} 0 \\ L \end{bmatrix} \Sigma^+ - Y_i(I - \Sigma\Sigma^+) \tag{3.78}$$

where $Y_i = \begin{bmatrix} Y_{i2} \\ Y_{i3} \end{bmatrix}$ is a matrix of appropriate dimensions with arbitrary elements. In this case, the particular solutions for S_i, K_{i2}, P_i and Q_i are given by

$$S_i = -Y_{i2}N_3 \tag{3.79}$$

$$M_i = S_iK_i - Y_{i2}\tilde{K}_3 \tag{3.80}$$

$$P_i = P_{11} - Y_3N_3 \tag{3.81}$$

$$Q_i = P_iK_i + \tilde{K}_{31} - Y_3\tilde{K}_3 \tag{3.82}$$

where $P_{11} = L\Sigma^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$ and $\tilde{K}_{31} = L\Sigma^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$.

We can see that the design is independent of the parameter matrices Y_{i3}, which can take an arbitrarily value. For simplicity we can take $Y_{i3} = 0$, in this case $P_1 = P_2 = P$.

Now, by using (3.74) and (3.79), the error dynamics (3.66) can be rewritten as

$$\dot{\vartheta}_i(t) = (\mathbb{A}_{i1} - \mathbb{Y}_i\mathbb{A}_2)\vartheta_i(t) \tag{3.83}$$

where $\mathbb{A}_{i1} = \begin{bmatrix} N_1 - K_iN_2 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{Y}_i = \begin{bmatrix} Y_{i1} & J_i \\ Y_{i2} & G_i \end{bmatrix}$ and $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I \end{bmatrix}$. The problem is now reduced to find matrices $\mathbb{Y}_i$ and K_i such that matrices $\mathbb{A}_i$ are Hurwitz. This can be reached by using any pole placement method which can be expressed in a linear matrix inequality (LMI) approach.

Now, the necessary and sufficient condition for the existence of matrices $\mathbb{Y}_i$ such that matrices $\mathbb{A}_i$ are Hurwitz is that the pair $(\mathbb{A}_2, \mathbb{A}_{i1})$ must be detectable.

The following Lemma gives the necessary and sufficient conditions for the detectability of the pair $(\mathbb{A}_2, \mathbb{A}_{i1})$.

Generalized Functional Dynamical Observer design

From the precedent results, the design of a generalized finite time observer is reduced to determine the parameters $\mathbb{Y}_i$ and K_i such that the eigenvalues of matrices $\mathbb{A}_i$ are inside two disks regions $\mathcal{Q}_1(\alpha_1, \tau_1)$ and $\mathcal{Q}_2(\alpha_2, \tau_2)$ by using Lemma 3.

The following theorem gives the sufficient conditions in LMIs formulation to realize this objective.

Theorem 2 *Under condition* (3.34), *let* $\mathcal{Q}_i(\alpha_i, \tau_i), i = 1, 2$ *be two disks region, then there exist parameters matrices* $\mathbb{Y}_i$ *and* K_i *such that the eigenvalues of matrices* $\mathbb{A}_i$ *are inside these regions, if there exists two symmetric positive definite matrices* $X_i = \begin{bmatrix} X_{i1} & X_{i1} \\ X_{i1} & X_{i2} \end{bmatrix} \in \mathbb{R}^{(q_0+q_1)\times(q_0+q_1)}$ *such that the following LMIs are satisfied:*

$$\begin{bmatrix} -X_i & \tilde{\mathbb{A}}_i \\ (*) & -N_3^{T\perp}X_{i1}N_3^{T\perp T} \end{bmatrix} < 0 \tag{3.84}$$

where $\tilde{\mathbb{A}}_i = \frac{1}{\tau_i}\begin{bmatrix} (X_{i1}N_1 - W_iN_2 - \alpha_iX_{i1})N_3^{T\perp T} \\ (X_{i1}N_1 - W_iN_2 - \alpha_iX_{i1})N_3^{T\perp T} \end{bmatrix}$, $W_i = X_{i1}K_i$ *and matrices* $\mathbb{Y}_i$ *can be parameterized as it follows:*

$$\mathbb{Y}_i = -\tau_i\left(\mathcal{B}_{ir}^+\mathcal{K}_i\mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_{ir}^+\mathcal{B}_{ir}\mathcal{Z}_i\mathcal{C}_l\mathcal{C}_l^+\right) \tag{3.85}$$

where

$$\mathcal{K}_i = -\mathcal{R}_i^{-1}\mathcal{B}_{il}^T\Theta_i\mathcal{C}_r^T(\mathcal{C}_r\Theta_i\mathcal{C}_r^T)^{-1} + \mathcal{S}_i^{1/2}\mathcal{L}_i(\mathcal{C}_r\Theta_i\mathcal{C}_r^T)^{-1/2} \tag{3.86}$$

$$\mathcal{S}_i = \mathcal{R}_i^{-1} - \mathcal{R}_i^{-1}\mathcal{B}_{il}^T[\Theta_i - \Theta_i\mathcal{C}_r^T(\mathcal{C}_r\Theta_i\mathcal{C}_r^T)^{-1}\mathcal{C}_r\Theta_i]\mathcal{B}_{il}\mathcal{R}_i^{-1} \tag{3.87}$$

and where $\mathcal{L}_i$ *is an arbitrary matrix such that* $||\mathcal{L}_i|| < 1$ *and* $\mathcal{R}_i$ *is an arbitrary positive definite matrix such that*

$$\Theta_i = (\mathcal{B}_{il}\mathcal{R}_i^{-1}\mathcal{B}_{il}^T - \mathcal{Q}_i)^{-1} > 0$$

for $i = 1, 2$.

Proof 4 *Under Assumption 1 and Condition* (3.34)*, from Lemma 3 the eigenvalues of matrices* $\mathbb{A}_i$ *are inside the two disks region* $\mathcal{Q}_i(\alpha_i, \tau_i), i = 1, 2,$ *if the following LMIs are satisfied*

$$\begin{bmatrix} -X_i & \frac{1}{\tau_i}(X_i\mathbb{A}_i - \alpha_i X_i) \\ (*) & -X_i \end{bmatrix} < 0 \tag{3.88}$$

where the symbol $(*)$ *denotes the transpose elements in the symmetric positions. Also* (3.88) *can be written as*

$$\mathcal{B}_i\mathcal{X}_i\mathcal{C} + (\mathcal{B}_i\mathcal{X}_i\mathcal{C})^T + \mathcal{Q}_i < 0 \tag{3.89}$$

where $\mathcal{B}_i = \begin{bmatrix} X_i \\ 0 \end{bmatrix}$, $\mathcal{X}_i = -\frac{1}{\tau_i}\mathbb{Y}_i$, $\mathcal{C} = \begin{bmatrix} 0 & \mathbb{A}_2 \end{bmatrix}$, *and* $\mathcal{Q}_i = \begin{bmatrix} -X_i & \frac{1}{\tau_i}(X_i\mathbb{A}_{i1} - \alpha_i X_i) \\ (*) & -X_i \end{bmatrix}$. *From Lemma 3, the solvability conditions of* (3.89) *are*

$$\mathcal{B}_i^{\perp}\mathcal{Q}_i\mathcal{B}_i^{\perp T} < 0 \tag{3.90}$$

$$\mathcal{C}^{T\perp}\mathcal{Q}_i\mathcal{C}^{T\perp T} < 0 \tag{3.91}$$

since $\mathcal{B}_i^{\perp} = \begin{bmatrix} 0 & I \end{bmatrix}$, *inequality* (3.90) *becomes* $X_i > 0$ *and considering* $\mathcal{C}^{T\perp} = \begin{bmatrix} I & 0 \\ 0 & \mathbb{A}_2^{T\perp} \end{bmatrix}$ *we obtain*

$$\begin{bmatrix} -X_i & \frac{1}{\tau_i}(X_i\mathbb{A}_{i1} - \alpha_i X_i)\mathbb{A}_2^{T\perp T} \\ (*) & -\mathbb{A}_2^{T\perp}X_i\mathbb{A}_2^{T\perp T} \end{bmatrix} < 0 \tag{3.92}$$

Now, it follows that

$$X_i\mathbb{A}_{i1} - \alpha_i X_i = \begin{bmatrix} X_{i1}N_1 - W_iN_2 - \alpha_i X_{i1} & -\alpha_i X_{i1} \\ X_{i1}N_1 - W_iN_2 - \alpha_i X_{i1} & -\alpha_i X_{i2} \end{bmatrix} \tag{3.93}$$

where $W_i = X_{i1}K_i$. *Since,* $\mathbb{A}_2^{T\perp} = \begin{bmatrix} N_3^{T\perp} & 0 \end{bmatrix}$, *then* $(X_i\mathbb{A}_{i1} - \alpha_i X_i)\mathbb{A}_2^{T\perp T}$ *can be written as*

$$(X_i\mathbb{A}_{i1} - \alpha_i X_i)\mathbb{A}_2^{T\perp T} = \begin{bmatrix} (X_{i1}N_1 - W_iN_2 - \alpha_i X_{i1})N_3^{T\perp T} \\ (X_{i1}N_{i1} - W_iN_2 - \alpha_i X_{i1})N_3^{T\perp T} \end{bmatrix}$$

Let $\tilde{\mathbb{A}}_i = \frac{1}{\tau_i}\begin{bmatrix} (X_{i1}N_1 - W_iN_2 - \alpha_i X_{i1})N_3^{T\perp T} \\ (X_{i1}N_1 - W_iN_2 - \alpha_i X_{i1})N_3^{T\perp T} \end{bmatrix}$, *then* (3.92) *can be written as*

$$\begin{bmatrix} -X_i & \tilde{\mathbb{A}}_i \\ \tilde{\mathbb{A}}_i^T & -N_3^{T\perp}X_{i1}N_3^{T\perp T} \end{bmatrix} < 0 \tag{3.94}$$

By using Lemma 7 we obtain the parameterization of matrix $\mathbb{Y}_i$ *as in* (3.85) *which completes the proof.*

Remark 6 *In order to achieve the pole placement of the observers, the pair* $(\mathbb{A}_2, \mathbb{A}_{i1})$ *must be observable*

Finite-time Convergent Observer

Now that we have two Functional Generalized Dynamical Observers of the form (3.60), we can design the finite-time convergent observer. Let $e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$ be the augmented estimation error consisting of the estimation error vectors from the FGDO's , $\hat{z}(t) = \begin{bmatrix} \hat{z}_1(t) \\ \hat{z}_2(t) \end{bmatrix}$ and $\vartheta(t) = \begin{bmatrix} \vartheta_1(t) \\ \vartheta_2(t) \end{bmatrix}$ then the augmented estimation error $e(t) = \hat{z}(t) - \Psi z(t)$, where $\Psi = \begin{bmatrix} I & I \end{bmatrix}$, can be written as

$$e(t) = \bar{P}\vartheta(t) \tag{3.95}$$

Where $\bar{P} = \begin{bmatrix} \tilde{P} & 0 \\ 0 & \tilde{P} \end{bmatrix}$ and $\tilde{P} = \begin{bmatrix} P & 0 \end{bmatrix}$

By solving Eq. (3.66), we have

$$\vartheta(t) = e^{\mathbb{A}(t-t_0)}\vartheta(t_0) \tag{3.96}$$

where $\mathbb{A} = \begin{bmatrix} \mathbb{A}_1 & 0 \\ 0 & \mathbb{A}_2 \end{bmatrix}$, and

$$\begin{aligned} \vartheta(t-h) &= e^{\mathbb{A}(t-h-t_0)}\vartheta(t_0) \\ \implies \vartheta(t_0) &= e^{-\mathbb{A}(t-h-t_0)}\vartheta(t-h) \end{aligned} \tag{3.97}$$

By substituting (3.97) in (3.96) we have

$$\vartheta(t) = e^{\mathbb{A}h}\vartheta(t-h) \tag{3.98}$$

Now we pre-multiply (3.98) by $\bar{P}$, then, using (3.95), we have

$$e(t) = \hat{z}(t) - \Psi z(t) = \bar{P}e^{\mathbb{A}h}\vartheta(t-h) \tag{3.99}$$

from (3.95), we have

$$\bar{P}\vartheta(t-h) = e(t-h) \tag{3.100}$$

which can be solved by

$$\vartheta(t-h) = \bar{P}^{+}e(t-h) + (I - \bar{P}^{+}\bar{P})\rho \tag{3.101}$$

where ρ is an arbitrary vector with appropriate dimension. For simplicity, we take $\rho = 0$, now we substitute (3.101) in (3.99)

$$\hat{z}(t) - \Psi z(t) = \bar{P}e^{\mathbb{A}h}\bar{P}^{+}\left(\hat{z}(t-h) - \Psi z(t-h)\right) \tag{3.102}$$

which can be rewritten as

$$\Psi z(t) - \bar{Q}\Psi z(t-h) = \hat{z}(t) - \bar{Q}\hat{z}(t-h) \tag{3.103}$$

where $\bar{Q} = \bar{P}e^{\mathbb{A}h}\bar{P}^{+}$ and

$$\begin{bmatrix}\Psi & \bar{Q}\Psi\end{bmatrix}\begin{bmatrix}z(t)\\ -z(t-h)\end{bmatrix} = \hat{z}(t) - \bar{Q}\hat{z}(t-h) \tag{3.104}$$

Then, $z(t)$ can be obtained by

$$\hat{z}(t) = \begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}\Psi & \bar{Q}\Psi\end{bmatrix}^{-1}\left(\hat{z}(t) - \bar{Q}\hat{z}(t-h)\right) \tag{3.105}$$

By making

$$T_f = \begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}\Psi & \bar{Q}\Psi\end{bmatrix}^{-1} \tag{3.106}$$

We can estimate the function by using

$$\hat{z}(t) = T_f\left(\hat{z}(t) - \bar{Q}\hat{z}(t-h)\right) \tag{3.107}$$

The existence of a finite-time convergence observer can be determined by matrix $\begin{bmatrix}\Psi & \bar{Q}\Psi\end{bmatrix}^{-1}$, in this case, this matrix exists if

$$\begin{aligned}\det\begin{bmatrix}\Psi & \bar{Q}\Psi\end{bmatrix} &\neq 0\\ \det\begin{bmatrix}I & \bar{Q}_1\\ I & \bar{Q}_2\end{bmatrix} &\neq 0\\ \det(\bar{Q}_1 - \bar{Q}_2) &\neq 0\end{aligned} \tag{3.108}$$

where $\bar{Q}_1 = \tilde{P}e^{\mathbb{A}_1 h}\tilde{P}^{+}$ and $\bar{Q}_2 = \tilde{P}e^{\mathbb{A}_2 h}\tilde{P}^{+}$.

Remark 7 *The necessary condition for* (3.108) *to be satisfied is reduced to* $\lambda(\mathbb{A}_1) < \rho < \lambda(\mathbb{A}_2)$

Numerical example

To illustrate the finite-time convergent functional generalized observer approach, consider an LTI system of the form (3.59) with:

$$A = \begin{bmatrix}-1.0 & 0 & 0 & 1.0 & 3.0\\ 2.0 & -5.0 & 0 & 1.0 & 2.0\\ 0 & 0 & -4.0 & 1.0 & 3.0\\ 1.0 & 0 & 0 & -2.0 & 4.0\\ 0 & 0 & 0 & -2.0 & -4.0\end{bmatrix},\ B = \begin{bmatrix}1 & 0\\ -1 & 0.5\\ 0 & 1\\ 0 & 0\end{bmatrix},\ C = \begin{bmatrix}1.0 & 0 & 0 & 1.0 & 0\\ 0 & 1.0 & 1.0 & 0 & 1.0\end{bmatrix},$$

and $L = \begin{bmatrix}1.0 & 0 & 0 & 0 & 0\\ 0 & 1.0 & 0 & 0 & 0\end{bmatrix}$

We choose $R = \begin{bmatrix}1.0 & 0 & 0 & 0 & 0\\ 0 & 1.0 & 0 & 0 & 0\\ 1.0 & 1.0 & 0 & 0 & 1.0\\ 1.0 & 0 & 1.0 & 0 & 0\end{bmatrix}$

Simulation 2 ***Finite-time convergent state estimator for an LTI system***

The objective of this simulation is to estimate the hidden state of a dynamical system in a given convergence time.

Settings. *We design two functional observers of the form* (3.60) *with their disks regions* $Q(\alpha_i, \tau_i)$ *chosen as* $\alpha_i = -7$, $\tau_1 = 2$, $\alpha_2 = -1$, $\tau_2 = 0.5$, *and a finite-time convergent estimator of the form* (3.107) *with a convergence time of* $0.5s$ *and initial condition* $x(0) = \begin{bmatrix} 2 & 1.5 & -1 & 0 & 1 \end{bmatrix}$. *The results are shown in Figure 3.4.*

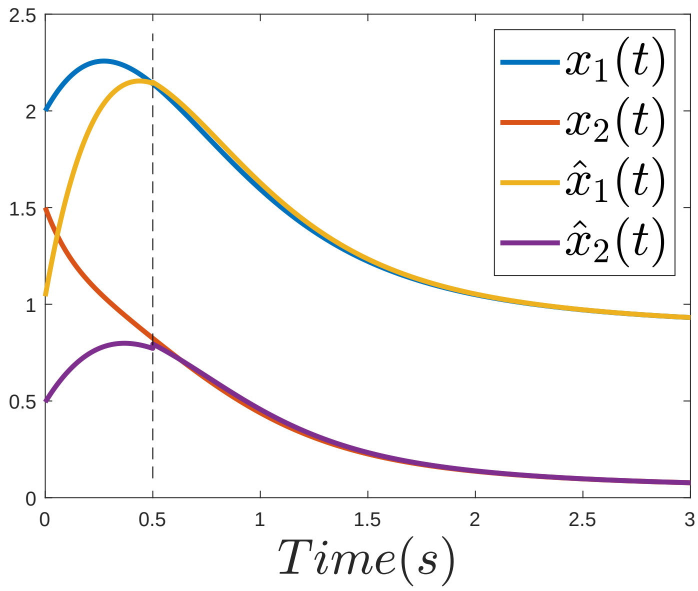


Figure 3.4: Estimated states

As we can see in Figure 3.4, the proposed functional observer is capable to estimate a partial state in the given convergence time.

3.2 Functional observer for descriptor systems with unknown input

This section presents a structure for a generalized functional observer prone to unknown inputs in the differential equations, for a class of descriptor linear systems of the form:

$$E\dot{x}(t) = Ax(t) + Bu(t) + Dd(t) \tag{3.109a}$$

$$y(t) = Cx(t) \tag{3.109b}$$

$$z(t) = Lx(t) \tag{3.109c}$$

where $x(t) \in \mathbb{R}^n$ is thestate of the system, $u(t) \in \mathbb{R}^m$ is the known input, $d(t) \in \mathbb{R}^l$ is the unknown input, $z(t) \in \mathbb{R}^q$ is the function of the state aimed to be estimated, and $y(t) \in \mathbb{R}^p$ is the measured output. This is an extension of the works made by [Osorio-Gordillo et al., 2016]. The necessary conditions for the existence and the determination of the observers gains are shown.

3.2.1 Problem statement

We consider the following generalized functional observer (GFO) of the form

$$\begin{aligned}
\dot{\zeta}(t) &= N\zeta(t) + Jv(t) + F\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} + Hu(t) + \Delta\hat{d}(t) \\
\dot{v}(t) &= S\zeta(t) + Gv(t) + M\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \\
\dot{\hat{d}}(t) &= \Phi\zeta(t) - Wy(t) \\
\hat{z}(t) &= P\zeta(t) + Q\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix}
\end{aligned} \tag{3.110}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ is the state of the observer, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector, $\hat{d}(t)$ is the estimate of the unknown input, and $\hat{z}(t) \in \mathbb{R}^q$ is the estimate of $z(t)$. N, J, F, H, Δ, S, G, M, Φ, W, P and Q are constant matrices of appropriate dimensions to be determined such that $\lim_{t\to\infty}(\hat{z}(t) - z(t)) = 0$.

The following Lemma gives the necessary conditions for the existence of the observer (3.110).

Lemma 9 *There exists an observer having the form given in (3.110) for the system (3.109) if the matrix*

$$\begin{bmatrix} N & J & \Delta \\ S & G & 0 \\ \Phi & 0 & 0 \end{bmatrix}$$

is Hurwitz and if there exists a matrix T such that the following conditions are satisfied:

a. $NTE + F\begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} - TA = 0$

b. $H = TB$

c. $\Delta = TD$

d. $STE + M\begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} = 0$

e. $\Phi TE - WC = 0$

f. $PTE + Q \begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} = L$

Proof 5 *Consider a parameter matrix* $T \in \mathbb{R}^{q_0 \times n}$ *and define the transformed error vector* $\varepsilon(t) = \zeta(t) - TEx(t)$. *Since* $E^{\perp}E = 0$ *and* $E^{\perp}D = 0$ *it can be deduced that* $E^{\perp}Ax(t) = -E^{\perp}Bu(t)$, *such that the derivative of* $\varepsilon(t)$ *is given by*

$$\dot{\varepsilon}(t) = N\zeta(t) + Jv(t) + \left(F \begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} - TA\right) x(t) + (H - TB)u(t) + \Delta\hat{d}(t) - TDd(t) \quad (3.111)$$

By using the definition of $\varepsilon(t)$, $\dot{v}(t)$ *in (3.60) can be rewritten as*

$$\dot{v}(t) = S\zeta(t) + Gv(t) + M\left(F \begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} - TA\right) x(t) \quad (3.112)$$

By considering the estimation error of the unknown input $\tilde{d}(t) = \hat{d}(t) - d(t)$, *the estimation error of the unknown input can be written as*

$$\dot{\tilde{d}}(t) = \Phi\zeta(t) - WCx(t) \quad (3.113)$$

Considering that $\dot{d}(t) = 0$ *and conditions a.-c. are satisfied, equations (3.111), (3.112) and* (3.113) *become*

$$\dot{\varepsilon}(t) = N\varepsilon(t) + Jv(t) + \Delta\tilde{d}(t) \quad (3.114)$$
$$\dot{v}(t) = S\varepsilon(t) + Gv(t) \quad (3.115)$$
$$\dot{\tilde{d}}(t) = \Phi\varepsilon(t) \quad (3.116)$$

By defining an augmented state vector $\sigma(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \\ \tilde{d}(t) \end{bmatrix}$, *equations* (3.114), (3.115) *can be rewritten as:*

$$\dot{\sigma}(t) = \mathbb{A}\sigma(t) \quad (3.117)$$

where

$$\mathbb{A} = \begin{bmatrix} N & J & \Delta \\ S & G & 0 \\ \Phi & 0 & 0 \end{bmatrix}$$

Define the estimation error $e(t) = z(t) - \hat{z}(t)$. *If condition c. is satisfied, then*

$$e(t) = P\varepsilon(t)$$

It can seen that if matrix $\mathbb{A}$ *is Hurwitz, then* $\lim_{t\to\infty} \varepsilon(t) = 0$ *and* $\lim_{t\to\infty} e(t) = 0$.

3.2.2 Observer parameterization

From Lemma 9, it can be deduced that the design of the observer is reduced to find the matrices N, J, F, H, Δ, S, G, M, Φ, W, P, Q and T such that conditions *a.-c.* are satisfied. We define now matrix $\Gamma = \begin{bmatrix} E \\ E^{\perp}A \\ C \end{bmatrix}$ and let $R \in \mathbb{R}^{q_0 \times n}$ be a full row rank matrix such that $rank \begin{bmatrix} R \\ \Gamma \end{bmatrix} = rank(\Gamma)$. There always exists two matrices T and K such that

$$TE + K \begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} = R \tag{3.118}$$

which can be rewritten as

$$\begin{bmatrix} T & K \end{bmatrix} \Gamma = R \tag{3.119}$$

the general solution for (3.119) is

$$\begin{bmatrix} T & K \end{bmatrix} = R\Gamma^{+} - Z\left(I - \Gamma\Gamma^{+}\right) \tag{3.120}$$

which can be decomposed in

$$T = T_1 - ZT_2 \tag{3.121}$$
$$K = K_1 - ZK_2 \tag{3.122}$$

where Z is a constant matrix of appropriate dimension, and

$$T_1 = R\Gamma^{+} \begin{bmatrix} I \\ 0 \end{bmatrix}$$
$$T_2 = \left(I - \Gamma\Gamma^{+}\right) \begin{bmatrix} I \\ 0 \end{bmatrix}$$
$$K_1 = R\Gamma^{+} \begin{bmatrix} 0 \\ I \end{bmatrix}$$
$$K_2 = \left(I - \Gamma\Gamma^{+}\right) \begin{bmatrix} 0 \\ I \end{bmatrix}$$

Now we define matrix $\Sigma = \begin{bmatrix} R \\ E^{\perp}A \\ C \end{bmatrix}$. From condition *a.* from Lemma 9 and (3.118), we have

$$N \left(R - K \begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} \right) + F \begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} = TA \tag{3.123}$$

which can be written as

$$\begin{bmatrix} N & \tilde{K} \end{bmatrix} \Sigma = TA \tag{3.124}$$

where $\tilde{K} = F - NK$. The necessary and sufficient condition for the existence of a solution to (3.124) is

$$rank \begin{bmatrix} \Sigma \\ TA \end{bmatrix} = rank(\Sigma)$$

the general solution to (3.124) is

$$\begin{bmatrix} N & \tilde{K} \end{bmatrix} = TA\Sigma^{+} - Y_1(I - \Sigma\Sigma^{+}) \tag{3.125}$$

if we replace (3.121) in (3.125), we obtain

$$N = N_1 - ZN_2 - Y_1N_3 \tag{3.126}$$
$$\tilde{K} = \tilde{K}_1 - Z\tilde{K}_2 - Y_1\tilde{K}_3 \tag{3.127}$$

where

$$N_1 = T_1A\Sigma^{+} \begin{bmatrix} I \\ 0 \end{bmatrix}$$
$$N_2 = T_2A\Sigma \begin{bmatrix} I \\ 0 \end{bmatrix}$$
$$N_3 = (I - \Sigma\Sigma^{+}) \begin{bmatrix} I \\ 0 \end{bmatrix}$$
$$\tilde{K}_1 = T_1A\Sigma^{+} \begin{bmatrix} 0 \\ I \end{bmatrix}$$
$$\tilde{K}_2 = T_2A\Sigma^{+} \begin{bmatrix} 0 \\ I \end{bmatrix}$$
$$\tilde{K}_3 = (I - \Sigma\Sigma^{+}) \begin{bmatrix} 0 \\ I \end{bmatrix}$$

since $F = \tilde{K} + NK$, we have

$$\begin{aligned} F &= \tilde{K}_1 + N_1K - Z(\tilde{K}_2 - N_2K) - Y_1(\tilde{K}_3 - N_3K) \\ F &= F_1 - ZF_2 - Y_1F_3 \end{aligned} \tag{3.128}$$

where

$$F_1 = T_1A\Sigma^{+} \begin{bmatrix} K \\ I \end{bmatrix}$$
$$F_2 = T_2A\Sigma^{+} \begin{bmatrix} K \\ I \end{bmatrix}$$
$$F_3 = (I - \Sigma\Sigma^{+}) \begin{bmatrix} K \\ I \end{bmatrix}$$

from (3.118) we have

$$\begin{bmatrix} TE \\ E^{\perp}A \\ C \end{bmatrix} = \begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} \Sigma \tag{3.129}$$

Conditions *c.* and *d.* of Lemma 9 can be written as

$$\begin{bmatrix} S & M \\ P & Q \end{bmatrix} \begin{bmatrix} TE \\ E^{\perp}A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ L \end{bmatrix} \tag{3.130}$$

if we substitute equation (3.129) into (3.130), we obtain

$$\begin{bmatrix} S & M \\ P & Q \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} \Sigma = \begin{bmatrix} 0 \\ L \end{bmatrix} \tag{3.131}$$

The necessary and sufficient condition for the existence of a solution to (3.131) is

$$rank \begin{bmatrix} \Sigma \\ L \end{bmatrix} = rank(\Sigma)$$

and since

$$\begin{bmatrix} I & -K \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & K \\ 0 & I \end{bmatrix}$$

the general solution is given by

$$\begin{bmatrix} S & M \\ P & Q \end{bmatrix} = \left(\begin{bmatrix} 0 \\ L \end{bmatrix} \Sigma^{+} - Y(I - \Sigma\Sigma^{+}) \right) \begin{bmatrix} I & K \\ 0 & I \end{bmatrix} \tag{3.132}$$

where $Y = \begin{bmatrix} Y_2 \\ Y_3 \end{bmatrix}$ is an arbitrary matrix of appropriate dimensions. In this case the particular solutions for S, M, P and Q are given by

$$S = -Y_2 N_3 \tag{3.133}$$
$$M = -Y_2 F_3 \tag{3.134}$$
$$P = P_1 - Y_3 N_3 \tag{3.135}$$
$$Q = Q_1 - Y_3 F_3 \tag{3.136}$$

where

$$P_1 = L\Sigma^{+} \begin{bmatrix} I \\ 0 \end{bmatrix}$$
$$Q_1 = L\Sigma^{+} \begin{bmatrix} K \\ I \end{bmatrix}.$$

Condition *e.* of Lemma 9 can be written as

$$\Phi \left(R - K \begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} \right) - WC = 0$$
$$\Phi R - \bar{K} \begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} = WC$$

$$\begin{bmatrix} \Phi & -\bar{K} \end{bmatrix} \Sigma = WC \tag{3.137}$$

The general solution for (3.137) is

$$\begin{bmatrix} \Phi & -\bar{K} \end{bmatrix} = WC\Sigma^{+} - Y_4(I - \Sigma\Sigma^{+}) \tag{3.138}$$

(3.138) can be rewritten as

$$\Phi = \Phi_1 - Y_4 N_3 \tag{3.139}$$
$$\bar{K} = \bar{K}_1 + Y_4 K_2 \tag{3.140}$$

where W is an arbitrary matrix of appropriate dimensions, $\Phi_1 = WC\Sigma^{+} \begin{bmatrix} I \\ 0 \end{bmatrix}$ and $\bar{K}1 = WC\Sigma^{+} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Now, by using (3.126), (3.133) and (3.139), the error dynamics (3.117) can be rewritten as

$$\dot{\sigma}(t) = (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\sigma(t) \tag{3.141}$$

where

$$\mathbb{A}_1 = \begin{bmatrix} N_1 - ZN_2 & 0 & \Delta \\ 0 & 0 & 0 \\ \Phi & 0 & 0 \end{bmatrix}, \ \mathbb{Y} = \begin{bmatrix} Y_1 & J & 0 \\ Y_2 & G & 0 \\ Y_4 & 0 & 0 \end{bmatrix} \text{ and } \mathbb{A}_2 = \begin{bmatrix} N_3 & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The problem has been now reduced to find matrices $\mathbb{Y}$ and Z such that the real parts of the eigenvalues of the matrix $\mathbb{A}$ are negative. This can be reached by using the linear matrix inequality (LMI) approach.

3.2.3 Observer design

The following theorem gives the LMI conditions that will allow to determine all the observer matrices.

Theorem 3.2.1 *Under Assumption 1 there exist two matrices $\mathbb{Y}$ and Z such that* (3.83) *is asymptotically stable if there exists a matrix*

$$X = \begin{bmatrix} X_1 & X_2 & X_3 \\ X_2^T & X_4 & X_5 \\ X_3^T & X_5^T & X_6 \end{bmatrix} > 0$$

such that the following LMI is satisfied:

$$N_3^{T\perp}\left(N_1^TX_1+X_1N_1-N_2^TW^T-WN_2+X_3\Phi_1+\Phi_1X_3^T\right)N_3^{T\perp T}<0. \tag{3.142}$$

Where $Z=X_1^{-1}W$, X_3 *is an arbitrary matrix of appropriate dimensions, and matrix* $\mathbb{Y}$ *is determined as follows*

$$\mathbb{Y}=-X^{-1}(-\gamma\mathcal{B}^T+\sqrt{\gamma}\mathcal{L}\Omega^{1/2})^T \tag{3.143}$$

where

$$\Omega=\gamma\mathcal{B}\mathcal{B}^T-\mathcal{Q}>0 \tag{3.144}$$

with

$$\mathcal{Q}=\begin{bmatrix} X_1(N_1-ZN_2)+(N_1-ZN_2)^TX_1+X_3\Phi_1+\Phi_1^TX_3^T & (N_1-ZN_2)^TX_2+\Phi_1^TX_5^T & X_1\Delta+(N_1-ZN_2)^TX_3+\Phi^TX_6 \\ X_2^T(N_1-ZN_2)+X_5\Phi_1 & 0 & X_2^T\Delta \\ X_3(N_1-ZN_2)+X_6\Phi+\Delta^TX_1 & 0 & X_3\Delta \end{bmatrix}$$

$$\mathcal{B}=\begin{bmatrix} N_3^T & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and matrix $\mathcal{L}$ *is any matrix such that* $\|\mathcal{L}\|<1$ *and* $\gamma>0$ *is any scalar such that* $\Omega>0$.

Proof 6 *We define the following Lyapunov function candidate*

$$V(\sigma(t))=\sigma(t)^TX\sigma(t) \tag{3.145}$$

its derivative is given by

$$\dot{V}(\sigma(t))=\sigma(t)^T\left[(\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2)^TX+X(\mathbb{A}_1-\mathbb{Y}\mathbb{A}_2)\right]\sigma(t) \tag{3.146}$$

The asymptotic stability of (3.141) *is guaranteed only if* $\dot{V}(\sigma(t))<0$, *this leas to the following LMI*

$$\mathbb{A}_1^TX-\mathbb{A}_2^T\mathbb{Y}^TX+X\mathbb{A}_1-X\mathbb{Y}\mathbb{A}_2<0 \tag{3.147}$$

which can be rewritten as

$$\mathcal{Q}+\mathcal{B}\mathcal{X}+(\mathcal{B}\mathcal{X})^T<0 \tag{3.148}$$

where $\mathcal{X}=-\mathbb{Y}^TX$, $\mathcal{Q}=X\mathbb{A}_1+\mathbb{A}_1^TX$ *and* $\mathcal{B}=\mathbb{A}_2^T$. *According to the Elimination Lemma, there exists a matrix* $\mathcal{X}$ *satisfying* (3.148) *if and only if the following condition holds*

$$\mathcal{B}^\perp\mathcal{Q}\mathcal{B}^{\perp T}<0 \tag{3.149}$$

with $\mathcal{B}^\perp=\begin{bmatrix}N_3^{T\perp} & 0 & 0\end{bmatrix}$. *By using the definitions of* $\mathcal{Q}$ *and* W, *we obtain* (3.142). *Matrix* $\mathbb{Y}$ *is obtained from* (3.143) *and* (3.144).

3.3 Functional observer for T-S descriptor systems

Consider a class of descriptor Takagi-Sugeno systems of the form:

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{k} w_i \left(A_i x(t) + B_i u(t)\right) \\ y(t) &= Cx(t) \end{aligned} \tag{3.150}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $u(t) \in \mathbb{R}^m$ is the input vector and $y(t) \in \mathbb{R}^p$ is the measured output of the system.

Assumption 2 *Descriptor system* (3.150) *is impulse observable, i.e.*

$$rank \begin{bmatrix} E \\ E^{\perp} A_i \\ C \end{bmatrix} = n \tag{3.151}$$

this is equivalent to $rank \begin{bmatrix} E & A_i \\ 0 & C \\ 0 & E \end{bmatrix} = rank(E) + n$

The following Lemma is presented in [Skelton et al., 1997] and will be used later.

Lemma 10 *Let matrices $\mathcal{B}$ and $\mathcal{Q}$ be given. The following statements are equivalent:*

1. *There exists a matrix $\mathcal{X}$ satisfying* $\mathcal{B}\mathcal{X} + (\mathcal{B}\mathcal{X})^T + \mathcal{Q} < 0$
2. *The following condition holds*

$$\mathcal{B}^{\perp} \mathcal{Q} \mathcal{B}^{\perp T} < 0$$

Suppose the above statements hold and assume that $\mathcal{B}^{\perp}\mathcal{B} > 0$. Then matrix $\mathcal{X}$ in statement 1 is given by

$$\mathcal{X} = -\gamma \mathcal{B}^T + \sqrt{\gamma}\mathcal{L}\Gamma^{1/2}$$

where $\mathcal{L}$ is any matrix such that $||\mathcal{L}|| < 1$ and $\gamma > 0$ is any scalar such that

$$\Gamma = \gamma \mathcal{B}\mathcal{B}^T - \mathcal{Q} > 0$$

3.3.1 Problem statement

Consider the following generalized functional observer (GFO) of the form

$$\begin{aligned} \dot{\zeta}(t) &= \sum_{i=1}^{k} w_i \left(N_i \zeta(t) + J_i v(t) + F_i y(t) + H_i u(t)\right) \\ \dot{v}(t) &= \sum_{i=1}^{k} w_i \left(S_i \zeta(t) + G_i v(t) + M_i y(t)\right) \\ \hat{z}(t) &= \sum_{i=1}^{k} w_i \left(P_i \zeta(t) + Q_i y(t)\right) \end{aligned} \tag{3.152}$$

where $\zeta(t) \in \mathbb{R}^{q_0}$ is the state of the observer, $v(t) \in \mathbb{R}^{q_1}$ is an auxiliary vector and $\hat{z}(t) \in \mathbb{R}^{q}$ is the estimate of $z(t)$. N_i, J_i, F_i, H_i, S_i, G_i, M_i, P_i and Q_i are constant matrices of appropriate dimensions to be determined such that $\lim_{t\to\infty} (\hat{z}(t) - z(t)) = 0$.
The following Lemma gives the sufficient conditions for the existence of the observer (3.152).

Lemma 11 *There exists an observer having the form given in (3.152) for the system (3.150) if the matrix* $\begin{bmatrix} N_i & J_i \\ S_i & G_i \end{bmatrix}$ *is Hurwitz and if there exists a matrix* T *such that the following conditions are satisfied:*

a. $N_i T E + F_i C - T A_i = 0$

b. $H_i = T B_i$

c. $S_i T E + M_i C = 0$

d. $P_i T E + Q_i C = L$

Proof 7 *Consider a parameter matrix* $T \in \mathbb{R}^{q_0 \times n}$ *and define the transformed error vector* $\varepsilon(t) = \zeta(t) - T E x(t)$. *Since* $E^{\perp} E = 0$ *it can be deduced that* $E^{\perp} A_i x(t) = -E^{\perp} B_i u(t)$, *such that the derivative of* $\varepsilon(t)$ *is given by*

$$\begin{aligned} \dot{\varepsilon}(t) = N_i \zeta(t) + (F_i C - T A_i)\, x(t) \\ +(H_i - T B_i) u(t) + J_i v(t) \end{aligned} \tag{3.153}$$

By using the definition of $\varepsilon(t)$, $\dot{v}(t)$ *in (3.152) can be rewritten as*

$$\dot{v}(t) = S_i \zeta(t) + M_i (F_i C - T A_i)\, x(t) + G_i v(t) \tag{3.154}$$

Considering that conditions a.-d. are satisfied, equations (3.153) and (3.154) become

$$\dot{\varepsilon}(t) = N_i \varepsilon(t) + J_i v(t) \tag{3.155}$$

$$\dot{v}(t) = S_i \varepsilon(t) + G_i v(t) \tag{3.156}$$

By defining an augmented state vector $\sigma(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}$, *equations* (3.155), (3.156) *can be rewritten as:*

$$\dot{\sigma}(t) = \mathbb{A}_i \sigma(t) \tag{3.157}$$

where $\mathbb{A}_i = \begin{bmatrix} N_i & J_i \\ S_i & G_i \end{bmatrix}$.
Define the estimation error $e(t) = z(t) - \hat{z}(t)$. *If condition c. is satisfied, then* $e(t) = P_i \varepsilon(t)$. *It can be seen that if matrix* $\mathbb{A}_i$ *is Hurwitz, then* $\lim_{t\to\infty} \varepsilon(t) = 0$ *and* $\lim_{t\to\infty} e(t) = 0$.

3.3.2 Observer parameterization

From Lemma 11, it can be deduced that the design of the observer is reduced to find the matrices N_i, J_i, F_i, H_i, S_i, G_i, M_i, P_i, Q_i and T such that conditions *a.-d.* are satisfied. We define now matrix $\Gamma = \begin{bmatrix} E \\ C \end{bmatrix}$ and let $R \in \mathbb{R}^{q_0 \times n}$ be a full row rank matrix such that $rank \begin{bmatrix} R \\ \Gamma \end{bmatrix} = rank(\Gamma)$. In this case there always exists two matrices T and K such that, $TE + KC = R$, which can be rewritten as

$$\begin{bmatrix} T & K \end{bmatrix} \Gamma = R \tag{3.158}$$

the general solution for (3.158) is

$$\begin{bmatrix} T & K \end{bmatrix} = R\Gamma^{+} - Z\left(I - \Gamma\Gamma^{+}\right) \tag{3.159}$$

which can be decomposed in

$$T = T_1 - Z_1 T_2 \tag{3.160}$$
$$K = K_1 - Z_1 K_2 \tag{3.161}$$

where Z is a constant matrix of appropriate dimension, and $T_1 = R\Gamma^{+} \begin{bmatrix} I \\ 0 \end{bmatrix}$,
$T_2 = (I - \Gamma\Gamma^{+}) \begin{bmatrix} I \\ 0 \end{bmatrix}$, $K_1 = R\Gamma^{+} \begin{bmatrix} 0 \\ I \end{bmatrix}$, and $K_2 = (I - \Gamma\Gamma^{+}) \begin{bmatrix} 0 \\ I \end{bmatrix}$.
Now we define matrix $\Sigma = \begin{bmatrix} R \\ C \end{bmatrix}$. From condition *a.* from Lemma 11 and (3.158), we have

$$N_i\left(R - KC\right) + F_i C = TA_i \tag{3.162}$$

which can be written as

$$\begin{bmatrix} N_i & \tilde{K}_i \end{bmatrix} \Sigma = TA_i \tag{3.163}$$

where $\tilde{K} = F_i - N_i K$. The necessary and sufficient condition for the existence of a solution to (3.163) is

$$rank \begin{bmatrix} \Sigma \\ TA_i \end{bmatrix} = rank(\Sigma)$$

the general solution to (3.163) is

$$\begin{bmatrix} N_i & \tilde{K}_i \end{bmatrix} = TA_i\Sigma^{+} - Y_{i1}(I - \Sigma\Sigma^{+}) \tag{3.164}$$

if we replace (3.160) in (3.164), we obtain

$$N_i = N_{i1} - ZN_{i2} - Y_{i1}N_3 \tag{3.165}$$
$$\tilde{K}_i = \tilde{K}_{i1} - Z\tilde{K}_{i2} - Y_{i1}\tilde{K}_{i3} \tag{3.166}$$

where $N_{i1} = T_1 A_i \Sigma^{+} \begin{bmatrix} I \\ 0 \end{bmatrix}$, $N_{i2} = T_2 A_i \Sigma \begin{bmatrix} I \\ 0 \end{bmatrix}$, $N_3 = (I - \Sigma\Sigma^{+}) \begin{bmatrix} I \\ 0 \end{bmatrix}$,, $\tilde{K}_{i1} = T_1 A_i \Sigma^{+} \begin{bmatrix} 0 \\ I \end{bmatrix}$,

$\tilde{K}_{i2} = T_2 A_i \Sigma^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$, $\tilde{K}_3 = (I - \Sigma\Sigma^+) \begin{bmatrix} 0 \\ I \end{bmatrix}$.

Since $F_i = \tilde{K}_i + N_i K$, we have

$$\begin{aligned} F_i &= \tilde{K}_{i1} + N_{i1} K - Z(\tilde{K}_{i2} - N_{i2} K) - Y_{i1}(\tilde{K}_3 - N_3 K) \\ F_i &= F_{i1} - Z F_{i2} - Y_{i1} F_3 \end{aligned} \tag{3.167}$$

where $F_{i1} = T_1 A_i \Sigma^+ \begin{bmatrix} K \\ I \end{bmatrix}$, $F_{i2} = T_2 A_i \Sigma^+ \begin{bmatrix} K \\ I \end{bmatrix}$, $F_{i3} = (I - \Sigma\Sigma^+) \begin{bmatrix} K \\ I \end{bmatrix}$.

From (3.158) we have

$$\begin{bmatrix} TE \\ C \end{bmatrix} = \begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} \Sigma \tag{3.168}$$

Conditions *c.* and *d.* of Lemma 11 can be written as

$$\begin{bmatrix} S_i & M_i \\ P_i & Q_i \end{bmatrix} \begin{bmatrix} TE \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ L \end{bmatrix} \tag{3.169}$$

replacing equation (3.168) into (3.169), we obtain

$$\begin{bmatrix} S_i & M_i \\ P_i & Q_i \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} \Sigma = \begin{bmatrix} 0 \\ L \end{bmatrix} \tag{3.170}$$

since $\begin{bmatrix} I & -K \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & K \\ 0 & I \end{bmatrix}$ the general solution is given by

$$\begin{bmatrix} S_i & M_i \\ P_i & Q_i \end{bmatrix} = \left(\begin{bmatrix} 0 \\ L \end{bmatrix} \Sigma^+ - Y_i (I - \Sigma\Sigma^+) \right) \begin{bmatrix} I & K \\ 0 & I \end{bmatrix} \tag{3.171}$$

where $Y_i = \begin{bmatrix} Y_{i2} \\ Y_{i3} \end{bmatrix}$ is an arbitrary matrix of appropriate dimensions. In this case the particular solutions for S_i, M_i, P_i and Q_i are given by

$$S_i = -Y_{i2} N_3 \tag{3.172}$$

$$M_i = -Y_{i2} F_3 \tag{3.173}$$

$$P_i = P_{i1} - Y_{i3} N_3 \tag{3.174}$$

$$Q_i = Q_{i1} - Y_{i3} F\_i3 \tag{3.175}$$

where $P_{i1} = L\Sigma^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$, $Q_{i1} = L\Sigma^+ \begin{bmatrix} K \\ I \end{bmatrix}$.

Now, by using (3.165) and (3.172), the error dynamics (3.157) it can be rewritten as

$$\dot{\sigma}(t) = (\mathbb{A}_{i1} - \mathbb{Y}_i \mathbb{A}_2)\sigma(t) \tag{3.176}$$

where $\mathbb{A}_{1i=} \begin{bmatrix} N_{i1} - Z N_{i2} & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{Y} = \begin{bmatrix} Y_{i1} & J_i \\ Y_{i2} & G_i \end{bmatrix}$ and $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I \end{bmatrix}$.

The problem has been now reduced to find matrices $\mathbb{Y}_i$ and Z such that matrix $\mathbb{A}_i = \mathbb{A}_{i1} - \mathbb{Y}_i \mathbb{A}_2$ is Hurwitz. This can be reached by using the linear matrix inequality (LMI) approach.

3.3.3 Functional observer design

Theorem 3.3.1 *Under Assumption 1 there exist two matrices* $\mathbb{Y}_i$ *and* Z *such that* (3.176) *is asymptotically stable if there exists a matrix*

$$X = \begin{bmatrix} X_1 & X_1 \\ X_1 & X_2 \end{bmatrix} > 0$$

such that the following LMI is satisfied:

$$N_3^{T\perp}\left(N_{i1}^T X_1 + X_1 N_{i1} - N_{i2}^T W^T - W N_{i2}\right) N_3^{T\perp T} < 0 \tag{3.177}$$

where $Z = X_1^{-1}W$ *and matrix* $\mathbb{Y}_i$ *can be determined as follows*

$$\mathbb{Y}_i = -X^{-1}(-\gamma \mathcal{B}^T + \sqrt{\gamma}\mathcal{L}\Omega_i^{1/2})^T \tag{3.178}$$

where

$$\Omega_i = \gamma \mathcal{B}\mathcal{B}^T - \mathcal{Q}_i > 0 \tag{3.179}$$

with

$$\mathcal{Q}_i = \begin{bmatrix} X_1(N_{i1} - ZN_{i2}) + (N_{i1} - ZN_{i2})^T X_1 & (N_{i1} - ZN_{i2}^T)X_1 \\ X_1(N_{i1} - ZN_{i2}) & 0 \end{bmatrix}$$

$$\mathcal{B} = \begin{bmatrix} N_3^T & 0 \\ 0 & -I \end{bmatrix}$$

and matrix $\mathcal{L}$ *is any matrix such that* $||\mathcal{L}|| < 1$ *and* $\gamma > 0$ *is any scalar such that* $\Omega_i > 0$*.*

Proof 8 *Consider the following Lyapunov function candidate*

$$V(\sigma(t)) = \sigma(t)^T X_i \sigma(t) \tag{3.180}$$

which derivative is given by

$$\dot{V}(\sigma(t)) = \sigma(t)^T\left[(\mathbb{A}_{i1} - \mathbb{Y}_i\mathbb{A}_2)^T X_i + X(\mathbb{A}_{i1} - \mathbb{Y}_i\mathbb{A}_2)\right]\sigma(t) \tag{3.181}$$

The asymptotic stability of (3.176) *is guaranteed only if* $\dot{V}(\sigma(t)) < 0$*, this leads to the following LMI*

$$\mathbb{A}_{i1}^T X - \mathbb{A}_2^T \mathbb{Y}_i^T X + X\mathbb{A}_{i1} - X\mathbb{Y}_i\mathbb{A}_2 < 0 \tag{3.182}$$

which can be rewritten as

$$\mathcal{Q}_i + \mathcal{B}\mathcal{X}_i + (\mathcal{B}\mathcal{X}_i)^T < 0 \tag{3.183}$$

where $\mathcal{X}_i = -\mathbb{Y}_i^T X$*,* $\mathcal{Q}_i = X\mathbb{A}_{i1} + \mathbb{A}_{i1}^T X$ *and* $\mathcal{B} = \mathbb{A}_2^T$*. According to Lemma 10, there exists a matrix* $\mathcal{X}_i$ *satisfying* (3.183) *if and only if the following condition holds*

$$\mathcal{B}^{\perp}\mathcal{Q}_i\mathcal{B}^{\perp T} < 0 \tag{3.184}$$

with $\mathcal{B}^{\perp} = \begin{bmatrix} N_3^{T\perp} & 0 \end{bmatrix}$*. By using the definitions of* $\mathcal{Q}_i$ *and* W*, we obtain* (3.177)*. Matrix* $\mathbb{Y}_i$ *is obtained from* (3.178)*.*

The following algorithm summarize the observer design to obtain the corresponding matrices.
Algorithm 1:

1. Choose a matrix R such that $rank \begin{bmatrix} R \\ \Gamma \end{bmatrix} = rank(\Gamma)$.
2. Compute matrices N_{i1}, N_{i2}, N_3, T_1, T_2, K_1, K_2, P_{i1} and Q_{i1}.
3. Solve the LMI (3.177) to find X and Z.
4. Choose a matrix $\mathcal{L}$ such that $||\mathcal{L}|| < 1$, and a scalar $\gamma > 0$ such that $\Omega_i > 0$, then determinate matrix $\mathbb{Y}_i$ as in (3.178).
5. Compute all the matrices gains of the observer (3.152) by using (3.165) to determinate N_i, (3.178) to determinate J_i and G_i, (3.172)-(3.175) to find S_i, M_i, P_i and Q_i taking matrix $Y_{i3} = 0$. F_i is given by (3.167) and matrix H_i could be determined with condition b.

3.3.4 Particular case: Proportional Functional observer

In order to obtain a Proportional Functional Observer (PFO) from the GFO, it corresponds to the parameter matrices $S = 0$, $J = 0$, $M = 0$, $G = 0$, $F_i = \begin{bmatrix} 0 & F_{ai} \end{bmatrix}$ and $Q_i = \begin{bmatrix} 0 & Q_{ai} \end{bmatrix}$, which generates the following observer [Trinh and Fernando, 2011]:

$$\begin{aligned} \dot{\zeta}(t) &= \sum_{i=1}^{k} w_i \left(N_i \zeta(t) + F_{ai} y(t) + H_i u(t) \right) \\ \hat{z}(t) &= \sum_{i=1}^{k} w_i \left(P_i \zeta(t) + Q_{ai} y(t) \right) \end{aligned} \tag{3.185}$$

and the error dynamics (3.157) becomes

$$\dot{\varepsilon}(t) = \left(\tilde{\mathbb{A}}_{i1} - \tilde{\mathbb{Y}}_i \tilde{\mathbb{A}}_2 \right) \varepsilon(t) \tag{3.186}$$

where $\tilde{\mathbb{A}}_{i1} = N_{i1} - ZN_{i2}$, $\tilde{\mathbb{Y}}_i = Y_{i1}$ and $\tilde{\mathbb{A}}_2 = N_3$. Matrices $\mathcal{Q}_i$, $\mathcal{B}$ and $\mathcal{X}_i$ of Theorem 1 become:

$$\begin{aligned} &\mathcal{Q}_i = X(N_{i1} - ZN_{i2}) + (N_{i1} - ZN_{i2})^T X, \\ &\mathcal{B} = N_3^T, \\ &\mathcal{X}_i = -Y_{i1}^T X \end{aligned}$$

Matrices Γ and Σ are defined as $\Gamma = \begin{bmatrix} E \\ C \end{bmatrix}$ and $\Sigma = \begin{bmatrix} R \\ C \end{bmatrix}$. With these matrices, the observer matrices can be obtained by the following the Algorithm 1.

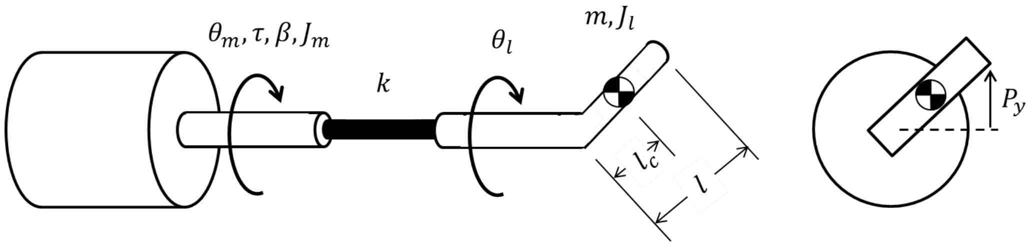


Figure 3.5: Single link robot arm

3.3.5 Mathematical model

The mathematical model chosen to test the performance of the generalized observer is a single link robot arm (see Fig. 3.5).

The model was selected from the one presented in [Chakrabarty et al., 2017] with the following nonlinear model:

$$
\begin{aligned}
J_m\ddot{\theta}_m(t) &= -\beta\dot{\theta}_m(t) - k(\theta_m(t) - \theta_l(t)) + K_\tau u(t) \\
J_l\ddot{\theta}_l(t) &= k(\theta_m(t) - \theta_l(t)) + l_c mg \sin\theta_l(t)
\end{aligned} \tag{3.187}
$$

The parameters of the model are given in Table 3.1

Table 3.1: Parameters of model

J_m	Moment of inertia of the motor shaft	0.01 $Kg * m^2$
J_l	Moment of inertia of the link	0.08 $Kg * m^2$
m	Mass of the link	0.15 m
l_c	Distance of the center of mass	0.25 m
k	Spring rigidity coefficient	0.3 Nm/rad
β	Viscous friction coefficient	0.05 Nms/rad
K_τ	Constant of the motor	0.2 Nm/V
l	Lenght of the link	0.18 m

Taking as the states $x_1(t) = \theta_m(t)$, $x_2(t) = \dot{\theta}_m(t)$, $x_3(t) = \theta_l(t)$, $x_4(t) = \dot{\theta}_l(t)$ and $x_5(t) = P_y(t)$, we can represent (2.6) as it follows:

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -\frac{\beta}{J_m}x_2(t) - \frac{k}{J_m}x_1(t) + \frac{k}{J_m}x_3(t) + \frac{K_\tau}{J_m}u(t) \\
\dot{x}_3(t) &= x_4(t) \\
\dot{x}_4(t) &= \frac{k}{J_l}x_1(t) - \frac{k}{J_l}x_3(t) + \frac{l_c mg}{J_l}\sin x_3(t) \\
0 &= l\sin x_3(t) - x_5(t)
\end{aligned} \tag{3.188}
$$

Takagi-Sugeno modeling

We chose as the scheduling variable $\rho(t) = \frac{\sin x_3(t)}{x_3(t)}$, so the system (3.188) can be represented:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{k}{J_m} & -\frac{\beta}{J_m} & \frac{k}{J_m} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{k}{J_l} & 0 & \frac{1}{J_m}\left(k - l_c m g \rho(t)\right) & 0 & 0 \\ 0 & 0 & l\rho(t) & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{K_\tau}{J_m} \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \tag{3.189}$$
$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} x(t)$$

The weighting functions for the T-S systems are:

$$w_1(t) = \frac{\bar{\rho} - \rho(t)}{\bar{\rho} - \underline{\rho}} \tag{3.190}$$
$$w_2(t) = 1 - w_1(t) \tag{3.191}$$

The Takagi-Sugeno system can be then represented as it follows:

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^{2} w_i(t)\left(A_i x(t) + B u(t)\right) \\ y(t) &= C x(t) \end{aligned} \tag{3.192}$$

with

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -30 & -5 & 30 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 3.75 & 0 & -8.33 & 0 & 0 \\ 0 & 0 & 0.18 & 0 & -1 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -30 & -5 & 30 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 3.75 & 0 & -2.77 & 0 & 0 \\ 0 & 0 & -0.04 & 0 & -1 \end{bmatrix}$$

3.3.6 Results

A generalized dynamical observer of the form:

$$\begin{aligned}\dot{\zeta}(t) &= \sum_{i=1}^{2} w_i \left(N_i\zeta(t) + J_i v(t) + F_i y(t) + H_i u(t) \right) \\ \dot{v}(t) &= \sum_{i=1}^{2} w_i \left(S_i\zeta(t) + G_i v(t) + M_i y(t) \right) \\ \hat{z}(t) &= \sum_{i=1}^{2} w_i \left(P_i\zeta(t) + Q_i y(t) \right)\end{aligned} \tag{3.193}$$

In order to have a comparison of the performance of the generalized dynamical observer, a proportional observer of the form

$$\begin{aligned}\dot{\zeta}(t) &= \sum_{i=1}^{2} w_i \left(N_{pi}\zeta(t) + F_{ai} y(t) + H_{pi} u(t) \right) \\ \hat{z}_p(t) &= \sum_{i=1}^{2} w_i \left(P_{pi}\zeta(t) + Q_{ai} y(t) \right)\end{aligned} \tag{3.194}$$

is designed for the same system. The matrix L is chosen in order to estimate the hidden state of the system.

$$L = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the matrix R

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The results of the simulation are shown in Fig. 3.6. The initial conditions for the nonlinear system are $x(0) = \begin{bmatrix} \frac{\pi}{12} & 0 & \frac{\pi}{6} & 0 & 0.09 \end{bmatrix}$ and the initial condition for both observers are $\mathbf{0}$. A parametric uncertainty of the form $\Delta k = 0.04\sin(t)$ is added to parameter k.

The matrix gains for the GDO are $N_1 = N_2 = \begin{bmatrix} -25.62 & 7.08 & 7.14 & 7.71 \\ -24.27 & -51.32 & -0.54 & -1.79 \\ -7.4 & -1.27 & -51.49 & -2.41 \\ -9.27 & 0.55 & 1.0 & -50.92 \end{bmatrix}$,

$J_1 = J_2 = \begin{bmatrix} 6.8 & 6.8 & 6.8 \\ 0.98 & 0.98 & 0.98 \\ 1.63 & 1.63 & 1.63 \\ 1.36 & 1.36 & 1.36 \end{bmatrix}$, $F_1 = \begin{bmatrix} 514.27 & 545.36 \\ -2234.93 & 149.92 \\ -67.01 & -2604.85 \\ 120.52 & 177.65 \end{bmatrix}$, $F_2 = \begin{bmatrix} 514.27 & 547.04 \\ -2234.93 & 149.53 \\ -67.01 & -2599.83 \\ 120.52 & 166.58 \end{bmatrix}$,

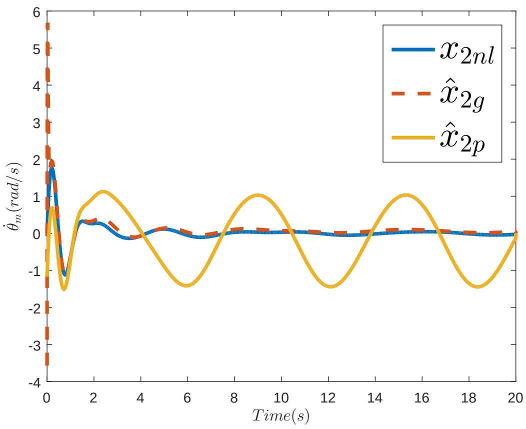
$H_1 = H_2 = \begin{bmatrix} 0 \\ 20 \\ 0 \\ 0 \end{bmatrix}$, $S_1 = S_2 = \begin{bmatrix} 9.91 & 0 & 0 & 0 \\ 9.91 & 0 & 0 & 0 \\ 9.91 & 0 & 0 & 0 \end{bmatrix}$, $G_1 = G2 = \begin{bmatrix} -28.77 & 3.93 & 3.93 \\ 3.93 & -28.77 & 3.93 \\ 3.93 & 3.93 & -28.77 \end{bmatrix}$,

$$M_1 = M_2 = \begin{bmatrix} -70.15 & -70.73 \\ -70.15 & -70.73 \\ -70.15 & -70.73 \end{bmatrix}, P_1 = P_2 = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}, Q_1 = Q_2 = \begin{bmatrix} 46.32 & 0.54 \\ 1.27 & 51.49 \\ -0.55 & -1.0 \end{bmatrix}.$$

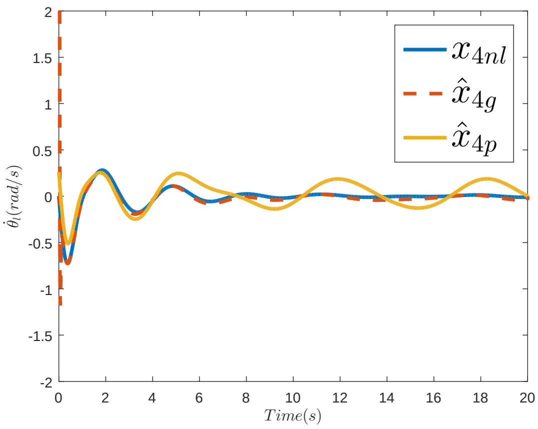
And the matrix gains for the proportional observer are $N_{p1} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}$,

$$N_{p2} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}, F_{a1} = \begin{bmatrix} -27.75 & 30.0 \\ 3.75 & -8.58 \\ 0 & 0.09 \end{bmatrix}, F_{a2} = \begin{bmatrix} -27.75 & 30.0 \\ 3.75 & -3.02 \\ 0 & -0.02 \end{bmatrix},$$

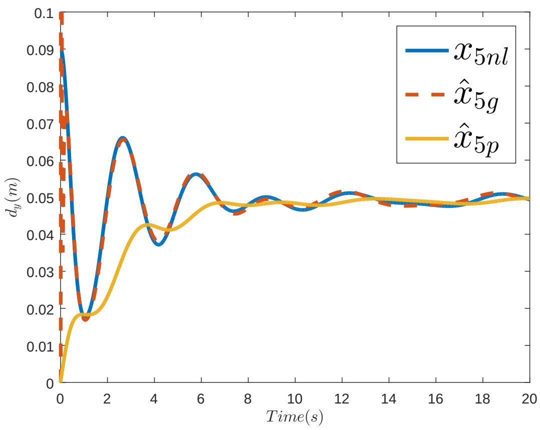
$$H_{p1} = H_{p2} = \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix}, P_{p1} = P_{p2} = \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, Q_{a1} = Q_{a2} = \begin{bmatrix} -4.5 & 0 \\ 0 & 0.5 \\ 0 & 0 \end{bmatrix}.$$



(a) $x_2(t)$ and its estimates



(b) $x_4(t)$ and its estimates



(c) $x_5(t)$ and its estimates

Figure 3.6: Results of the simulation.

As it can be seen in Figs. 3.6, the generalized observer is capable to estimate the state of the system even in presence of parametric uncertainties. This is an advantage compared with the classical proportional observers.

3.3.7 Application to state and fault estimation for LTI systems

In this section we will apply the obtained results to the estimation of faults in the state equation. Consider the following descriptor system affected by faults

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + F_a f(t) \\ y(t) &= Cx(t) + F_s f(t) \\ z(t) &= Lx(t) \end{aligned} \tag{3.195}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $u(t) \in \mathbb{R}^m$ is the input of the system, $f(t) \in \mathbb{R}^{n_f}$ is the fault, $y(t) \in \mathbb{R}^p$ is the measured output, and $z(t) \in \mathbb{R}^q$ is a function of the states to be estimated. E, A, B, C, L, F_a and F_s are constant known matrices. The objective is to design a finite-time functional observer for simultaneous state and fault estimation for system (3.195). Before presenting the main results of this section, we can make the following remarks.

Remark 8 *The model (3.195) is written in a general form. In fact if the actuator fault is different of the sensor one, they can be included in (3.195). Let $f_a(t)$ and $f_s(t)$ be the actuator and sensor faults respectively, then it is sufficient to take $f(t) = \begin{bmatrix} f_a(t) \\ f_s(t) \end{bmatrix}$ and modify the model (3.195) according to this partition of the faults.*

Remark 9 *Concerning the assumption on the fault $f(t)$, we assume that $f(t)$ is constant or piecewise constant. This approach can also be used for time-varying faults by taking the second derivative of $f(t)$ equal to zero and by augmenting the state of the system according to this assumption.*

By augmenting the state $x(t)$ with the fault vector $f(t)$, the system (3.195) can be written as

$$\begin{aligned} \mathbf{E}\dot{\chi}(t) &= \mathbf{A}\chi(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\chi(t) \\ \breve{z}(t) &= \mathbf{L}\chi(t) \end{aligned} \tag{3.196}$$

where $\chi(t) = \begin{bmatrix} x(t) \\ f(t) \end{bmatrix}$, $\breve{z}(t) = \begin{bmatrix} z(t) \\ f(t) \end{bmatrix}$, $\mathbf{E} = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} A & F_a \\ 0 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$ $\mathbf{C} = \begin{bmatrix} C & F_s \end{bmatrix}$ and $\mathbf{L} = \begin{bmatrix} L & 0 \\ 0 & I \end{bmatrix}$.

For this system, we can see that Assumption 1 can be written as

$$\operatorname{rank} \begin{bmatrix} L \\ E \\ E^{\perp}A \\ C \end{bmatrix} = \operatorname{rank} \begin{bmatrix} L & 0 \\ E & 0 \\ 0 & I \\ E^{\perp}A & E^{\perp}F_a \\ C & F_a \end{bmatrix} = \operatorname{rank} \begin{bmatrix} L \\ E \\ E^{\perp}A \\ C \end{bmatrix} + n_f$$

and

$$\operatorname{rank}\begin{bmatrix} E \\ E^{\perp}A \\ C \end{bmatrix} = \operatorname{rank}\begin{bmatrix} E & 0 \\ 0 & I \\ E^{\perp}A & E^{\perp}F_a \\ C & F_a \end{bmatrix} = \operatorname{rank}\begin{bmatrix} E \\ E^{\perp}A \\ C \end{bmatrix} + n_f$$

where we have the fact that $C^{+}A^{+} = \begin{bmatrix} E^{\perp}A & E^{\perp}F_a \end{bmatrix}$.

Now, the design of the observer can be achieved by using the method described above. It can be seen that, by choosing matrix $\mathbf{L}$, the functional observer allows to estimate the faults as well as the desired states of the augmented system.

The following algorithm can be used to obtain the different gain matrices in order to fulfill the functional observer based controller for System 3.196.
Algorithm 1.

1. *Choose matrices* R_i *such that* $\operatorname{rank}\begin{bmatrix} R_i \\ \Gamma_i \end{bmatrix} = \operatorname{rank}(\Gamma_i)$.
2. *Compute matrices* T_{1i}, T_{2i}, K_{1i}, K_{2i}, N_{1i}, N_{2i}, and N_{3i} *defined in section* IV.
3. *Solve LMI* (3.177) *to obtain* X, *and* Z_i.
4. *Choose matrices* $\mathcal{L}_i$ *such that* $||\mathcal{L}_i|| < 1$, *and matrices* $\mathcal{R}_i$ *such that* $\Phi_i > 0$. Obtain $\mathbb{Y}_i$ using (3.178).
5. *Compute all the matrices of the* FGDO *(3.152) by using* (3.160) *to obtain* T_i, (3.165) *to obtain* N_i, (3.167) *to obtain* F_i, (3.172)-(3.175) *to obtain* S_i, M_i, P_i, *and* Q_i, *respectively and* H_i *by using* $H_i = T_iB$.

Numerical example

The generalized approach of the proposed observer is less susceptible to parametric uncertainties, compared to the proportional observer. To illustrate this, consider the following uncertain descriptor system affected by faults.

$$\begin{aligned} E\dot{x}(t) &= (A + \Delta A)x(t) + Bu(t) + F_a f(t) \\ y(t) &= Cx(t) + F_s f(t) \end{aligned} \tag{3.197}$$

with $E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 0.66 & 1.7 & 7.9 & 0.42 & -3.05 & -1.01 \\ 0.95 & -2.44 & 2.19 & 1.99 & -1.77 & 3.56 \\ -0.26 & -0.19 & -3.06 & -0.03 & 0.17 & -0.18 \\ 2.47 & 1.84 & 9.28 & -1.54 & -2.96 & -0.39 \\ 3.83 & 1.91 & 12.64 & 2.18 & -7.6 & 1.32 \\ -1.51 & -0.39 & -2.8 & -0.73 & 1.35 & -3.02 \end{bmatrix}$, $\Delta A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\delta(t)$

$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$, $F_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $F_s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $L = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$, and

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The uncertainty variation $\delta(t)$ is illustrated in Figure 3.7.

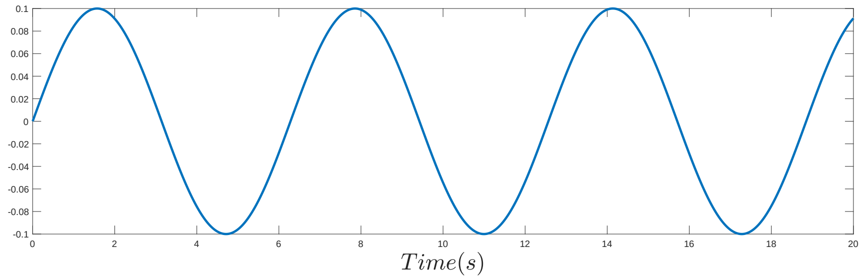


Figure 3.7: Uncertainty variation

Simulation 3 ***Finite-time Generalized Dynamic Observer performance comparison with a Proportional Observer.***

The objective of this simulation is to evaluate the performance of the Finite-time Generalized Dynamic Observer in presence of parametric uncertainties in the system and compare it to a Proportional Observer. For this system, we design a Finite-time observer based on the equations (3.152) *simultaneously estimating the state and the fault, we also design the finite-time Proportional observer (see remark 2).*

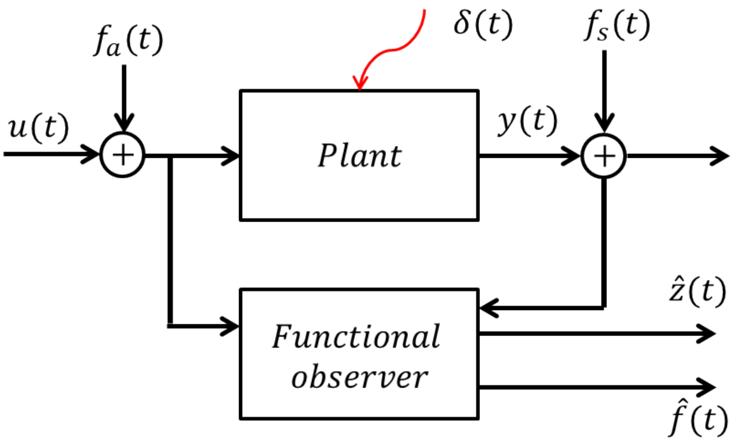


Figure 3.8: Block diagram of the functional observer

It can be seen in Figures 3.9 - 3.10, that the proposed functional observer estimates the function of the state and the fault in the given convergence time, and reduces the effects of

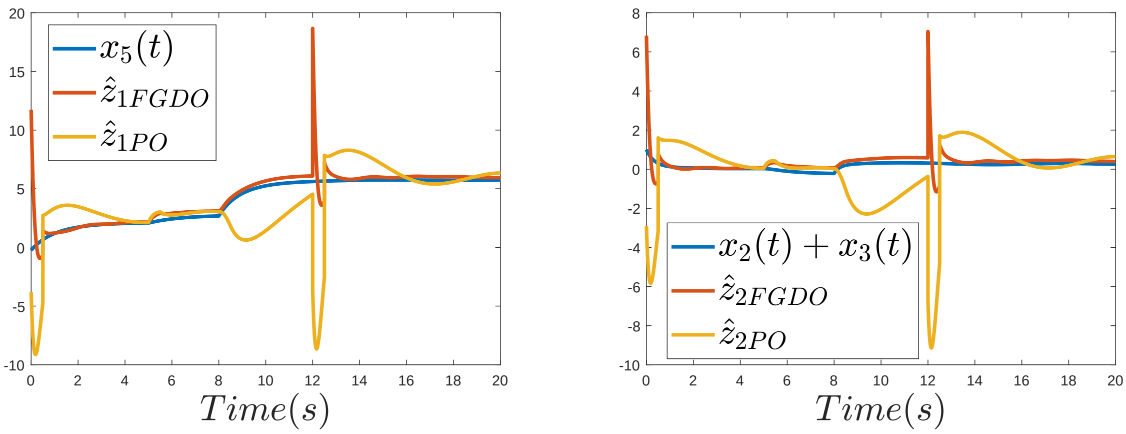


Figure 3.9: Function estimation

the parametric uncertainties in comparison to the PO. The comparison between the different observers can be seen in Table 3.2 for the Integral of the Absolute Error, and Table 3.3 for the Integral of the Square Error.

Table 3.2: IAE performance index

	$z_1(t)$	$z_2(t)$	$f_1(t)$	$f_2(t)$	$f_3(t)$
FGDO	8.33	4.88	8.35	24.21	4.81
PO	34.4	21.99	6.91	160.68	5.67

Table 3.3: ISE performance index

	$z_1(t)$	$z_2(t)$	$f_1(t)$	$f_2(t)$	$f_3(t)$
FGDO	19.35	5.15	30.39	131.86	15.92
PO	163.58	69.98	271	4 097.1	192

3.4 Conclusions

In conclusion, the synthesis of generalized functional observers explored in this Chapter introduces a versatile framework that enables the design of observers with dimensions distinct from the original system. This flexibility proves instrumental in crafting reduced-order observers for estimating control laws, offering practical advantages in system optimization. By incorporating

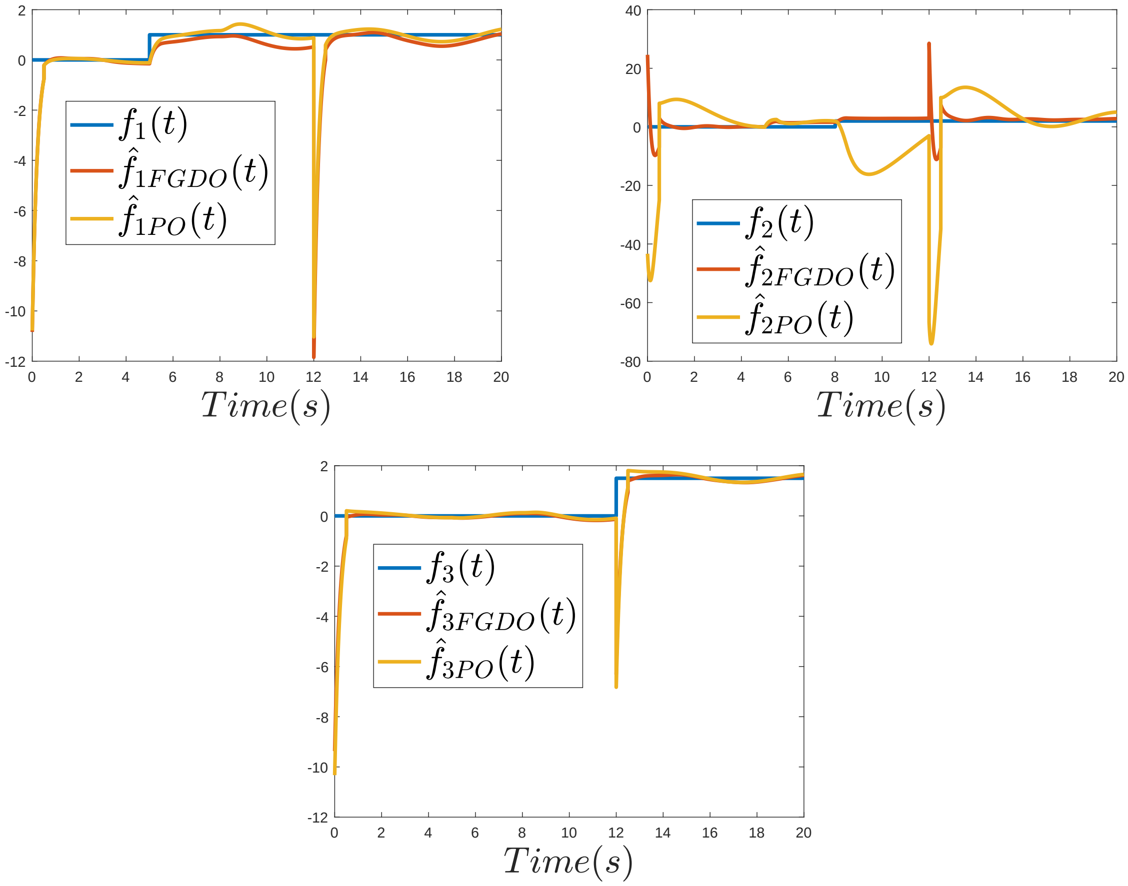


Figure 3.10: Fault estimation

augmented states that encompass faults, the methodology extends its utility to the creation of fault-tolerant control systems, particularly through the implementation of reduced-order Functional observers.

Furthermore, the generalized approach demonstrated resilience to parametric uncertainties, contrasting favorably with Proportional or Proportional-Integral observers, as evidenced in the numerical example provided. The outlined stability and existence conditions contribute to the theoretical underpinning of this approach, establishing a solid foundation for its application. The numerical example serves as a tangible illustration, showcasing the fault estimator's robust capabilities. This chapter not only advances the theoretical understanding of generalized functional observers but also highlights their practical implications for control system design and fault tolerance strategies.

In addition to its application in generalized functional observers, the synthesis approach extends seamlessly to Takagi-Sugeno modeling and linear time-invariant systems with unknown inputs. The adaptation of the framework to Takagi-Sugeno models broadens its scope, allowing for the effective handling of complex, nonlinear systems. This flexibility proves invaluable in scenarios where traditional linear models fall short, offering a more accurate representation of real-world dynamics.

Moreover, when applied to linear time-invariant systems with unknown inputs, the synthesized observers showcase their robustness by providing accurate estimations even in the presence of uncertainties. The ability to address unknown inputs enhances the practical relevance of the methodology, especially in situations where disturbances or external influences may affect system behavior. The numerical examples presented in this chapter serve as compelling evidence of the efficacy of the proposed synthesis approach in handling the complexities associated with Takagi-Sugeno models and systems with unknown inputs. This dual extension underscores the versatility and applicability of the generalized functional observer framework across a spectrum of challenging control scenarios.

Chapter 4

Application of functional observers

Contents

This chapter serves as a comprehensive exploration of observer-based control within the context of the generalized approach for functional observers. Observer-based control, a cornerstone of modern engineering, plays a vital role in achieving precise regulation and manipulation of dynamic processes. By accurately estimating the internal states of a system, engineers can develop control strategies that are robust, responsive, and versatile. The objective of this chapter is to provide a thorough understanding of observer-based control, its fundamental principles, design methodologies, and practical applications.

Through the exploration of functional observers, this chapter aims to provide a comprehensive understanding of their design methodologies, including the necessary and sufficient conditions for their existence.

Additionally, this chapter extends its coverage to the integration of fault-tolerant control within the framework of observer-based control. Fault tolerance ensures the system's robustness and resilience by detecting and accommodating faults that may arise during operation. The chapter explores fault detection, identification, and accommodation techniques, highlighting their integration with observer-based control to achieve simultaneous fault-tolerant control and state estimation.

4.1 Functional observer-based control

Let us consider the following LTI descriptor system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ z(t) &= Lx(t) \end{aligned} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input control vector, $y(t) \in \mathbb{R}^p$ is the measured output, and $z(t) \in \mathbb{R}^q$ is the function to estimate. E, A, B, C, and L are constant known matrices.

4.1.1 Problem statement

The problem is to design a control law of the form $u(t) = Lx(t) + u_r(t)$, where $u_r(t)$ is the reference. In this case, system 4.1 becomes:

$$\begin{aligned} E\dot{x}(t) &= (A + BL)x(t) + Bu_r(t) \\ y(t) &= Cx(t) \end{aligned} \tag{4.2}$$

Now, we can use the functional observer to estimate the functional $z(t) = -Lx(t)$. To find this estimate we will used the generalized observer proposed in Chapters 2 and 3.

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + Jv(t) + F\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} + Hu(t) \\ \dot{v}(t) &= S\zeta(t) + Gv(t) + M\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \\ \hat{z}(t) &= P\zeta(t) + Q\begin{bmatrix} -E^{\perp}Bu(t) \\ y(t) \end{bmatrix} \end{aligned} \tag{4.3}$$

The control system scheme can be seen in Figure 4.1.1.

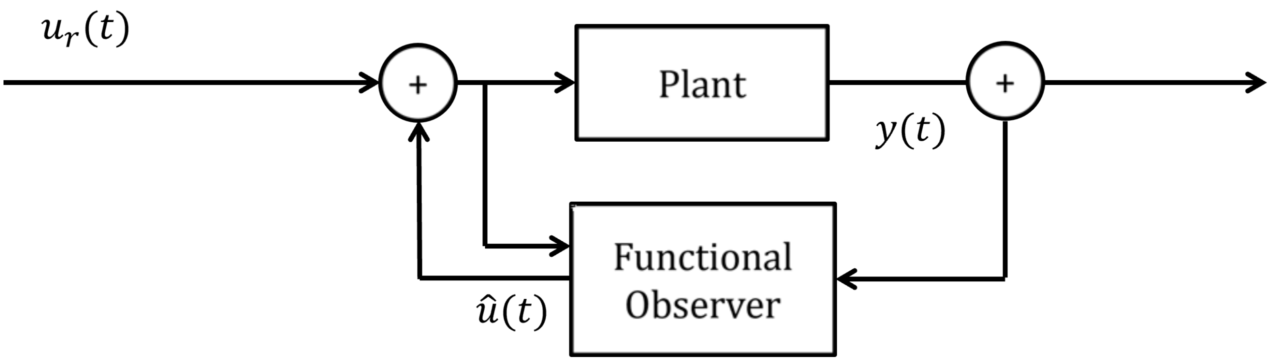


Figure 4.1: Block diagram for the observer-based control system

Then, by using the assumption that

$$rank \begin{bmatrix} L \\ E \\ E^{\perp}A \\ C \end{bmatrix} = rank \begin{bmatrix} E \\ E^{\perp}A \\ C \end{bmatrix}$$

given in Chapter 3, and by following the approach presented in Chapter 3, we define the following transformed error.

$$\varepsilon(t) = \zeta(t) - TEx(t)$$

to obtain

$$\begin{aligned} \dot{\varepsilon}(t) &= N\varepsilon(t) + J(t) \\ \dot{v}(t) &= S\varepsilon(t) + Gv(t) \end{aligned} \tag{4.4}$$

Lets define the augmented error vector $\sigma(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}$, then its dynamics are represented as it follows

$$\dot{\sigma}(t) = \mathbb{A}\sigma(t) \tag{4.5}$$

where $\mathbb{A} = \begin{bmatrix} N & J \\ S & G \end{bmatrix}$.

Now, since $e(t) = \hat{z}(t) - z(t) = \hat{z}(t) - Lx(t)$, then

$$e(t) = P\varepsilon(t) \tag{4.6}$$

from (4.6), it can be seen that $\lim_{t\to\infty} e(t) = 0$ if $\lim_{t\to\infty} \varepsilon(t) = 0$ the stability is given when matrix $\mathbb{A}$ is stable with the constraints

a. $NTE + F \begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} - TA = 0$

b. $STE + M \begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} = 0$

c. $PTE + Q \begin{bmatrix} E^{\perp}A \\ C \end{bmatrix} = L$

d. $H = TB$

The parameters of the observer can be obtained as presented in the previous Chapters.

4.1.2 Controller design

Now, the controller can be obtained by combining systems (4.1) and (4.3) in this case, based on the constraints a., b., c., and d. we obtain

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{4.7}$$
$$u(t) = \hat{z}(t) + u_r(t) \tag{4.8}$$

if we substitute (4.8) into (4.7), we obtain

$$E\dot{x}(t) = Ax(t) + B\hat{z}(t) + Bu_r(t) \tag{4.9}$$

from (4.6), we have

$$\hat{z}(t) = P\varepsilon(t) + Lx(t) \tag{4.10}$$

(4.4) with (4.9) leads to

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + BP\varepsilon(t) + BLx(t) + Bu_r(t) \\ \dot{\varepsilon}(t) &= N\varepsilon(t) + J(t) \\ \dot{v}(t) &= S\varepsilon(t) + Sv(t) \end{aligned} \tag{4.11}$$

or equivalently

$$\begin{aligned} E\dot{x}(t) &= (A + BL)x(t) + BP\varepsilon(t) + Bu_r(t) \\ \dot{\varepsilon}(t) &= N\varepsilon(t) + J(t) \\ \dot{v}(t) &= S\varepsilon(t) + Sv(t) \end{aligned} \tag{4.12}$$

which can be rewritten as

$$\mathcal{E}\dot{\bar{x}}(t) = \mathcal{A}\bar{x}(t) + \mathcal{B}u_r(t) \tag{4.13}$$

where $\bar{x}(t) = \begin{bmatrix} x(t) \\ \varepsilon(t) \\ v(t) \end{bmatrix}$ is an augmented vector, $\mathcal{E} = \begin{bmatrix} E & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$, $\mathcal{A} = \begin{bmatrix} A + BL & BP & 0 \\ 0 & N & J \\ 0 & S & G \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}$.

It can be seen from (4.13) that the system is stable if, and only if the pair $(\mathcal{E}, \mathcal{A},)$ is stable.

From equation (4.13), we can see that the eigenvalues of the closed loop system can be obtained from $\mathcal{A} = 0$, which shows that the eigenvalues of the closed loop system are the union between those of $(E, A + BL)$ and those of $\begin{bmatrix} N & J \\ S & G \end{bmatrix}$.

On the other hand, the determination of the observer parameters are related to the matrix L obtained from the pair $(E, A + BL)$. Then, the design of the observer-based control can be obtained in two steps.

- Find a matrix L, such that the pair $(E, A + BL)$ is stable.
- Find the matrices $\begin{bmatrix} N & J \\ S & G \end{bmatrix}$ of the observer based on the obtained L matrix.

The following theorem gives the condition for the admissibility of (4.9).

Theorem 4.1.1 *System* (4.9) *is said to be regular, impulse free and stable if and only if, there exist a matrix P, and a matrix $X = \begin{bmatrix} X_1 & X_1 \\ X_1 & X_2 \end{bmatrix} > 0$ such that*

$$EP = P^T E^T \tag{4.14}$$

$$(A + BL)P + P^T (A + BL)^T < 0 \tag{4.15}$$

and

$$\begin{bmatrix} -X & \tilde{\mathbb{A}} \\ (*) & \mathbb{A}_1^{T\perp} X \mathbb{A}_2^{T\perp T} \end{bmatrix} < 0 \tag{4.16}$$

where $\tilde{\mathbb{A}} = \frac{1}{\tau} \begin{bmatrix} (X_1 N_1 - W N_2 - \alpha X_1) N_3^{T\perp T} \\ (X_1 N_1 - W N_2 - \alpha X_1) N_3^{T\perp T} \end{bmatrix}$, $\mathbb{A}_1 = \begin{bmatrix} N_1 - Z N_2 & 0 \\ 0 & 0 \end{bmatrix}$, *and* $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I \end{bmatrix}$, *following the parameterization presented in Chapter 2, where* $\mathbb{A} = \begin{bmatrix} N & J \\ S & G \end{bmatrix}$.

Proof 9 *For the pair $(E, A + LB)$ to be admissible if and only if there exists a matrix P_c such that*

$$EP_c = P_c^T E^T \geq 0 \tag{4.17}$$

and

$$(A + BL)P_c + P_c^T (A + BL)^T < 0 \tag{4.18}$$

(4.18) can be rewritten as

$$AP_c + BLP_c + P_c^T A + P_c^T L^T A^T < 0 \tag{4.19}$$

In order to avoid the bilinearity problem, let $W_L = LP_c$, then $W_L^T = P_c^T L^T$, this leads to

$$AP_c + BW_L + P_c^T A + W_L^T A^T < 0 \tag{4.20}$$

4.1.3 Numerical example

To illustrate this approach, we propose the following numerical example. For a system of the form (4.1), with:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -3.0 & 2.0 & 1.0 & 1.0 \\ 1.0 & -2.0 & 1.0 & -2.0 \\ 3.0 & 0 & 1.0 & -0.5 \\ -1.0 & -2.0 & 0 & 1.0 \end{bmatrix}, B = \begin{bmatrix} 1.0 & 0 \\ 0.5 & -1.0 \\ 0 & 1.0 \\ 0 & 0 \end{bmatrix}, \text{and } C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalues of the pair (E, A) are $\lambda = \begin{bmatrix} 1.79 & -4.4 - 1.72\mathrm{i} & -4.4 + 1.72\mathrm{i} \end{bmatrix}^T$, from this, it can be seen that the descriptor system is unstable. In order to stabilize it, we need to find a matrix L that makes the system admissible.

A suitable Matrix is found by using the Yalmip toolbox for Matlab. In this case, the designated stabilizing matrix is

$$L = \begin{bmatrix} -8.95 & -26.07 & 12.1 & -4.85 \\ 1.71 & 1.31 & -3.51 & -0.16 \end{bmatrix}$$

and the eigenvalues of the pair $(E, A+BL)$ are $\lambda = \begin{bmatrix} -1.7 + 1.99\mathrm{i} & -1.7 - 1.99\mathrm{i} & -39.78 \end{bmatrix}^T$

Now we design a functional observer of the form (4.3) to estimate the function of the state $z(t) = Lx(t)$.

By using the method proposed in Chapter 3, we propose a generalized functional observer with the poles placed in a $\mathcal{D}$-region with its center in $\alpha = -2$ and a radius $\tau = 1$. The matrices of the observer are

$$N = \begin{bmatrix} -3.24 & -4.26 \\ -0.1 & -2.33 \end{bmatrix}, J = \begin{bmatrix} 2.94 & 2.94 \\ 0.3 & 0.3 \end{bmatrix}, F = \begin{bmatrix} 8.12 & 39.76 & -15.39 & 17.1 \\ -2.01 & -11.6 & 5.58 & -2.45 \end{bmatrix},$$

$$H = \begin{bmatrix} -25.04 & 34.63 \\ 7.74 & -9.09 \end{bmatrix}, S = \begin{bmatrix} 0.2 & 0.7 \\ 0.11 & 0.37 \end{bmatrix}, G = \begin{bmatrix} -2.73 & -0.73 \\ -0.38 & -2.38 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.05 & -0.62 & 0.76 & -0.05 \\ 0.09 & -0.26 & 0.54 & -0.09 \end{bmatrix}, P = \begin{bmatrix} 1.0 & -0.01 \\ -0.01 & 0.96 \end{bmatrix},$$

and $Q = \begin{bmatrix} -0.37 & 4.48 & -4.29 & -4.48 \\ 1.67 & -1.53 & -0.98 & -1.83 \end{bmatrix}$.

with these parameter matrices and controller L, the eigenvalues of the pair $(\mathcal{E}, \mathcal{A})$ are $\begin{bmatrix} -1.7 + 1.99\mathrm{i} & -1.7 - 1.99\mathrm{i} & -39.78 & -2.0 & -2.0 & -2.2 & -4.47 \end{bmatrix}^T$ which makes the augmented system controller-observer admissible.

Simulation

The simulation of the Observer-based controller is done with Simulink. The initial values of the system are $x(0) = \begin{bmatrix} 16 & -1 & 3 & 7 \end{bmatrix}^T$. $u_r(t)$ is set to zero. The results are shown in Figures 4.1.3 and 4.1.3.

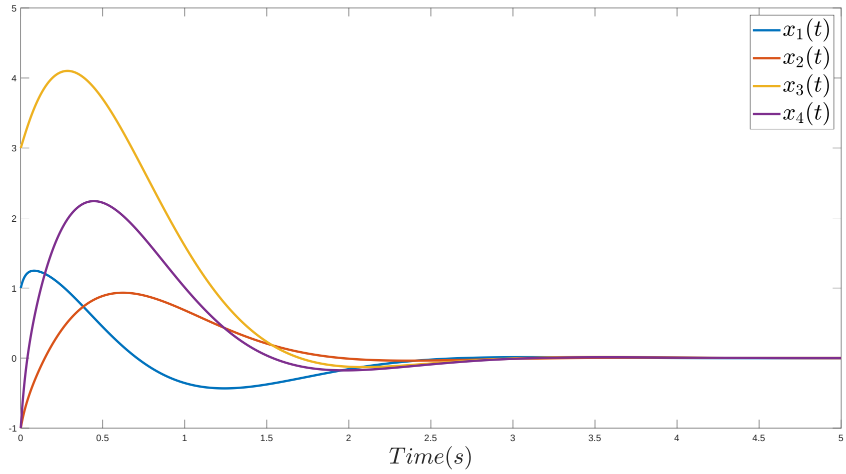


Figure 4.2: States of the system

It can be seen in Figure 4.1.3 that the states converge to 0, the observer is able to stabilize the descriptor system.

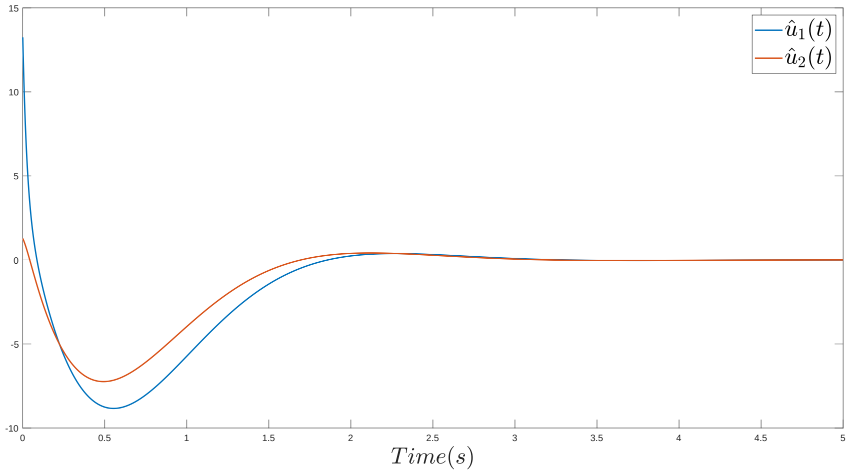


Figure 4.3: Estimated control law

A functional observer can be used as a controller for unstable descriptor LTI systems, this can be advantageous in applications where the computational cost has a great impact on the performance on the control system. For example, a functional observer can be used in small insulin pumps, where embedded systems don't have the full capability of a computer.

On the other hand, the generalized approach of the functional observer allows to mitigate the effects on parametric uncertainties to a certain extend.

4.2 Sensor fault-tolerant control

The FTC scheme for sensor faults is illustrated in Figure 4.4

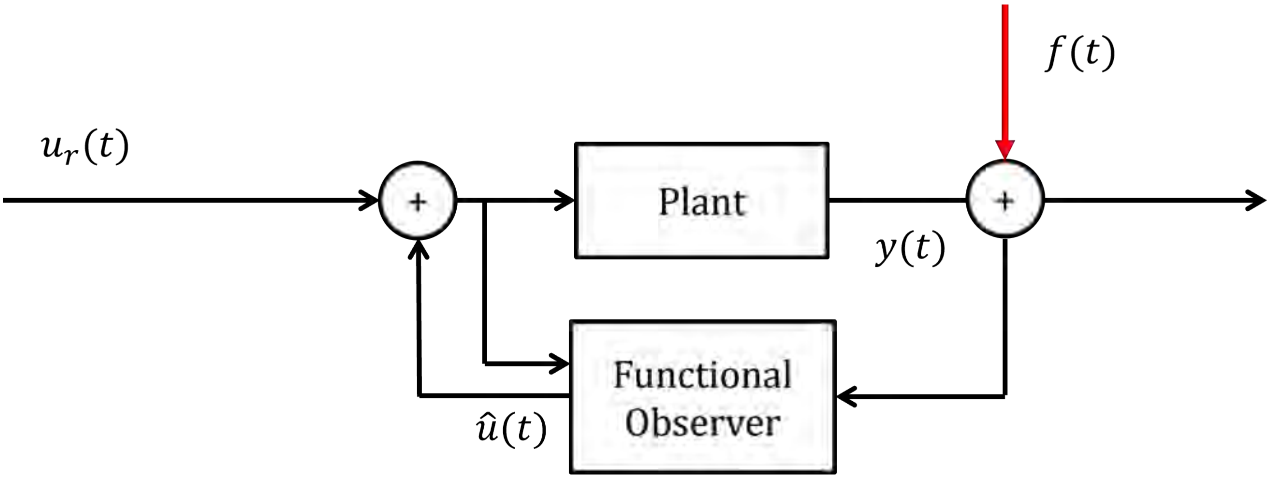


Figure 4.4: Block Diagram for sensor a FTC system

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input control vector, $y(t) \in \mathbb{R}^p$ is the measured output, and $\hat{u}(t) \in \mathbb{R}^m$ is the estimate of $u(t)$.

4.2.1 Problem statement

The LTI Descriptor system is described by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Ff(t) \\ z(t) &= Lx(t) \end{aligned} \tag{4.21}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $F \in \mathbb{R}^{n\times n_f}$, $C \in \mathbb{R}^{p\times n}$, and $L \in \mathbb{R}^{q\times n}$ are known constant matrices.

For the FTC we make the following assumption

Assumption 3 *The faults in the system are constant, their dynamics are described by*

$$\dot{f}(t) = 0$$

Let $\bar{x}(t) = \begin{bmatrix} x(t) & f(t) \end{bmatrix}^T$ be an augmented state containing both, the state of the system, and the faults. The dynamics of $\bar{x}(t)$ are

$$\begin{aligned} \bar{E}\dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}\hat{u}(t) \\ y(t) &= \bar{C}\bar{x}(t) \\ \hat{z}(t) = \hat{u}(t) &= \bar{L}\bar{x}(t) \end{aligned} \tag{4.22}$$

where

$$\bar{E} = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix},\ \bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},\ \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix},\ \bar{C} = \begin{bmatrix} C & F \end{bmatrix},\ \text{and } \bar{L} = \begin{bmatrix} L & 0 \end{bmatrix}$$

In this case, the controller design follows the same process as depicted in section 4.1. Now, it can be seen that system (4.21) is stable if the pair $(\bar{E}, \bar{A})$ is stable. This can be verified by

$$\begin{gathered} \bar{E}\bar{P}_c = \bar{P}_c^T \bar{E}^T \geq 0 \\ (\bar{A} + \bar{B}\bar{L})\bar{P}_c + \bar{P}_c^T(\bar{A} + \bar{B}\bar{L})^T < 0 \end{gathered} \tag{4.23}$$

4.2.2 Numerical example

For a system of the form (4.21) with:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} -3 & 2 & 1 \\ -2 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0.5 & -1 \\ 0 & 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \text{and } F_s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

we propose to design a fault-tolerant functional observer-based controller. The estimaed control law is $\hat{u}(t) = \begin{bmatrix} 2.61 & 1.09 & -1.38 \\ -0.43 & 0.04 & 0.02 \end{bmatrix} x(t)$. Then, the augmented $\bar{L}$ is

$$\bar{L} = \begin{bmatrix} 2.61 & 1.09 & -1.38 & 0 & 0 \\ -0.43 & 0.04 & 0.02 & 0 & 0 \end{bmatrix}$$

It can be seen that this matrix $\bar{L}$ does not take into account the faults for the control law, the estimated control law is then $\hat{z}(t) = Lx(t)$, hence making the system fault-tolerant.

The functional observer of the form (4.2) has the following parameter matrices:

$$N = \begin{bmatrix} -4.92 & 0.26 \\ 0.08 & -4.74 \end{bmatrix},\ J = \begin{bmatrix} 0.22 & 0.22 \\ 0.22 & 0.22 \end{bmatrix},\ F = \begin{bmatrix} 3.43 & -0.46 & -5.0 \\ -1.37 & -1.27 & 1.09 \end{bmatrix},$$

$$S = \begin{bmatrix} -0.0511 & -0.1585 \\ -0.0064 & -0.0200 \end{bmatrix},\ G = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix},\ M = \begin{bmatrix} 0.0166 & -0.4619 & -0.339 \\ 0.0021 & -0.0582 & -0.0427 \end{bmatrix},$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \text{and } Q = \begin{bmatrix} -0.366 & -0.1064 & -1.0110 \\ -0.1906 & -0.2696 & 0.2076 \end{bmatrix}$$

Simulation

The simulation of the Observer-based controller is done with Simulink. The initial values of the system are $x(0) = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}^T$.

The sensor faults are illustrated in Figure 4.5

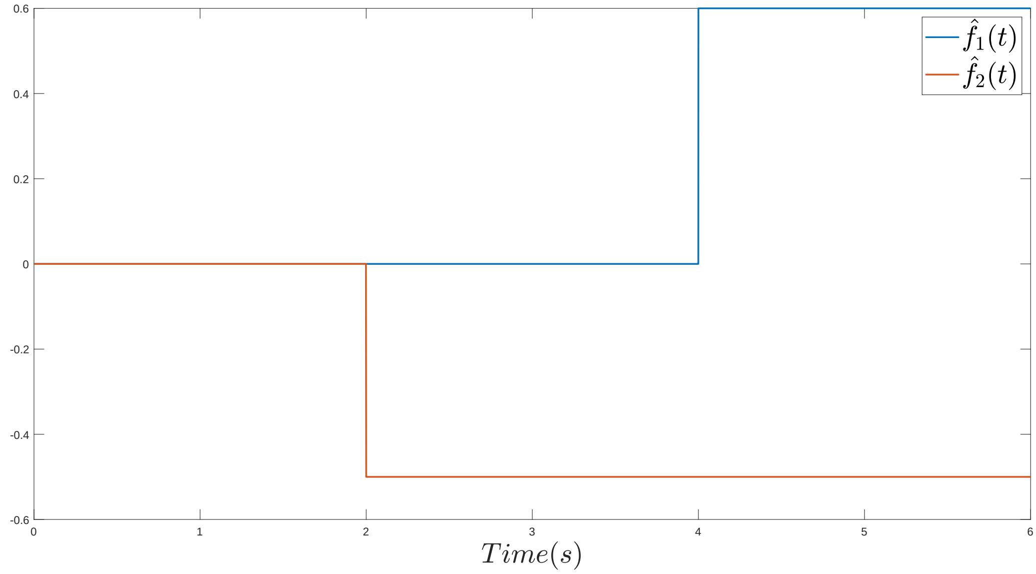


Figure 4.5: Sensor fault

The results are shown in Figures 4.6-4.8.

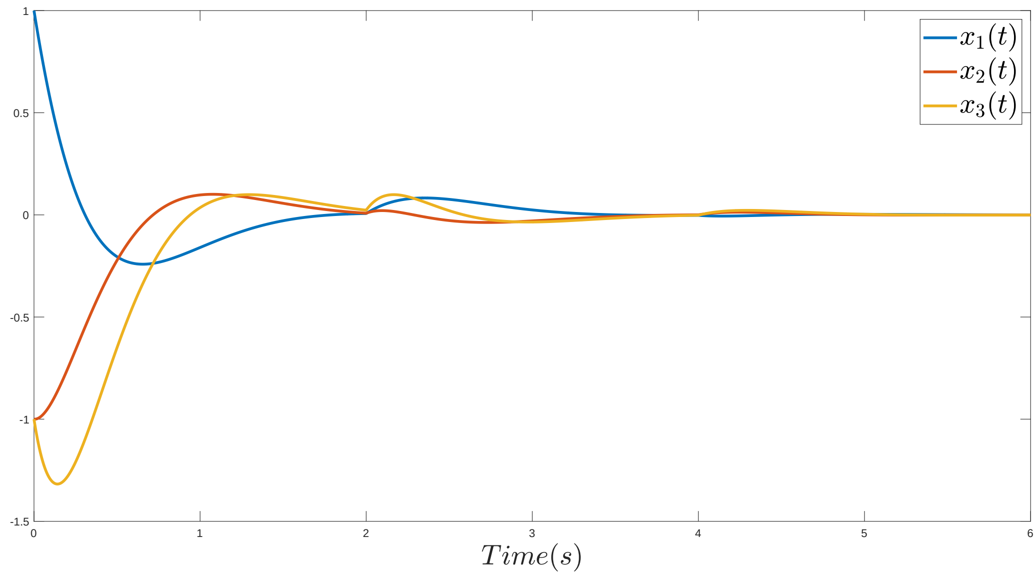


Figure 4.6: States of the system

As we can see in Figure 4.6, the states of the system are stabilized even in the presence of the sensor faults.

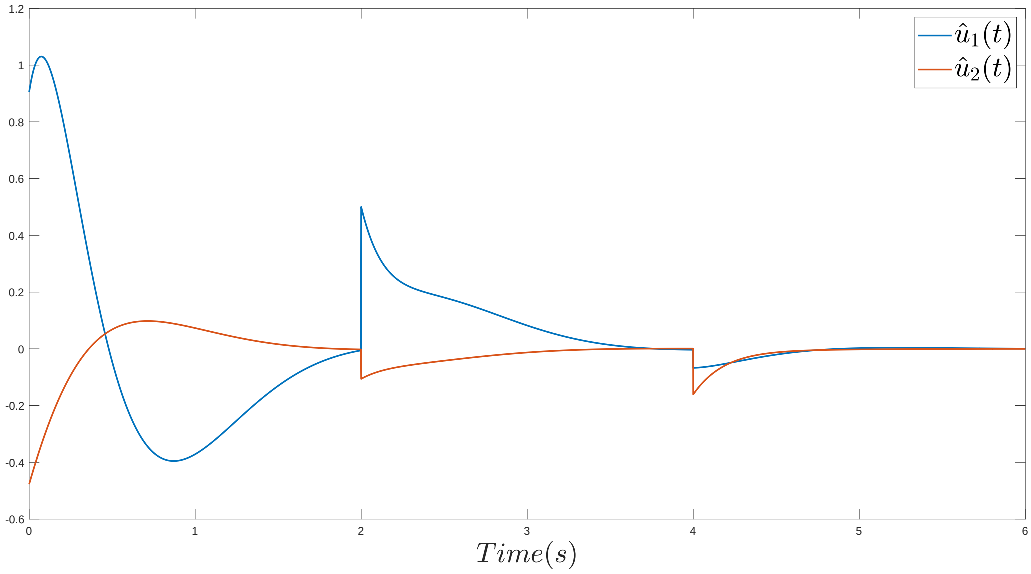


Figure 4.7: Estimated control law

The control law responds to the appearance and adapts itself to stabilize the system. In comparison, Figure 4.8 illustrate the effects of a control law $u(t) = Lx(t)$ that is not fault-tolerant.

A functional observer can be used as a controller for descriptor LTI systems prone to sensor faults. The augmented system allows the observer to correctly estimate the function of the state while mitigating the effects of the faults in the system.

4.3 Actuator fault-tolerant control

The active FTC scheme used in this section is illustrated in Figure 4.9

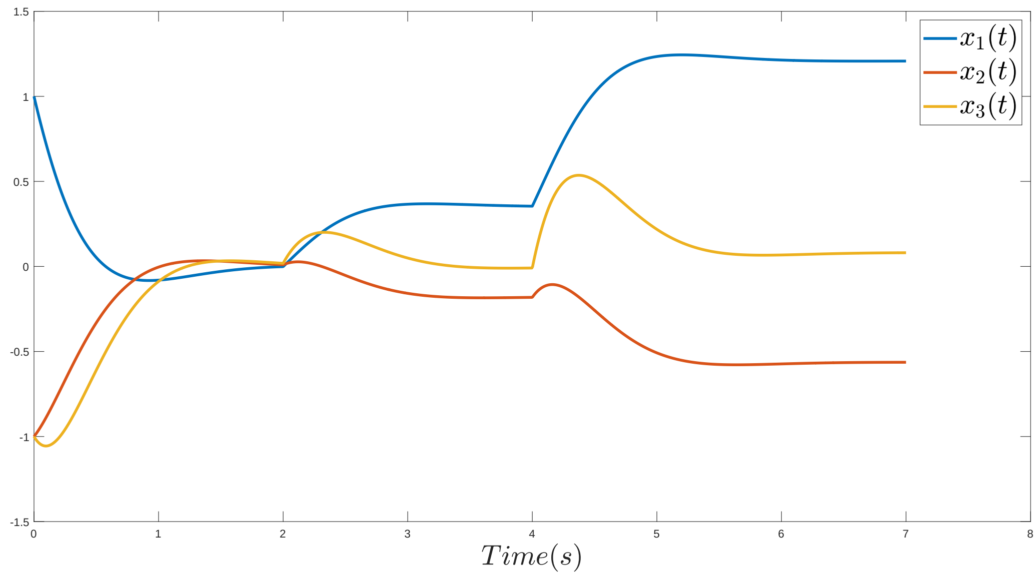


Figure 4.8: States of the system

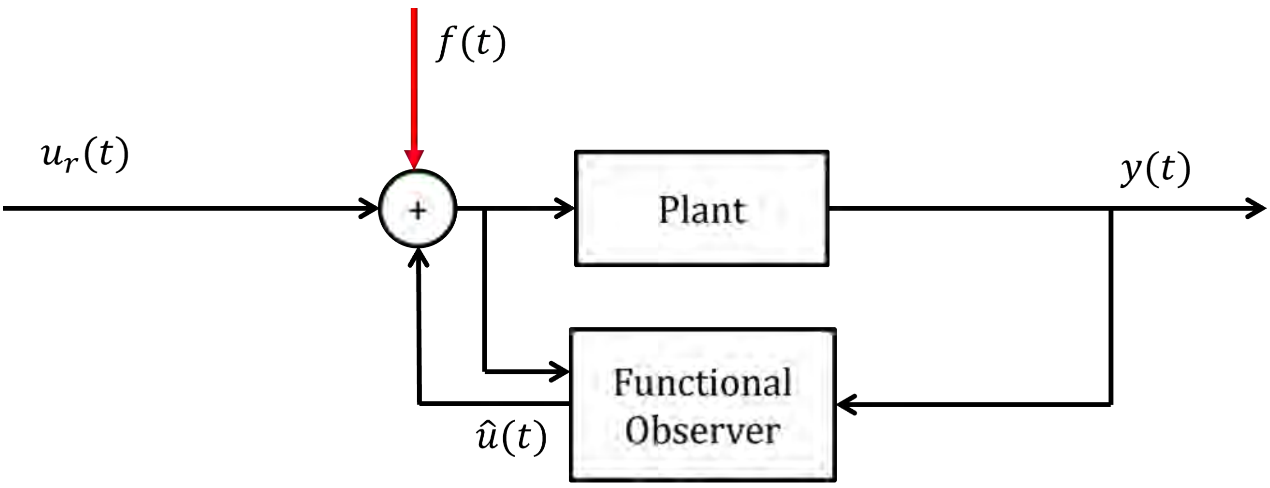


Figure 4.9: Fault-tolerant control scheme

where $x(t) \in \mathbb{R}^n$ is the state vector, $u_c(t) \in \mathbb{R}^m$ is the input control vector, $y(t) \in \mathbb{R}^p$ is the measured output, and $\hat{u}(t) \in \mathbb{R}^m$ is the estimate of $u_c(t)$.

4.3.1 Problem statement

The LTI Descriptor system is described by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + B\left(u(t) + f(t)\right) \\ y(t) &= Cx(t) \\ z(t) &= Lx(t) \end{aligned} \tag{4.24}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $L \in \mathbb{R}^{q\times n}$ are known constant matrices.

The FTC design is based on a functional observer that mitigates the effects of actuator faults occurring on he system. The FTC control law is given by

$$u(t) = Lx(t) - f(t) + u_r(t) \tag{4.25}$$

For the FTC design we make the following assumption

Assumption 4 *The faults in the system are constant, their dynamics are described by*

$$\dot{f}(t) = 0$$

Let $\bar{x}(t) = \begin{bmatrix} x(t) f(t) \end{bmatrix}^T$ be an augmented state containing both, the state of the system, and the faults. The dynamics of $\bar{x}(t)$ are

$$\begin{aligned} \bar{E}\dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}\hat{u}(t) \\ y(t) &= \bar{C}\bar{x}(t) \\ \hat{u}(t) &= \bar{L}\bar{x}(t) \end{aligned} \tag{4.26}$$

where

$$\bar{E} = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix},\ \bar{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix},\ \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix},\ \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \text{ and } \bar{L} = \begin{bmatrix} L & -I \end{bmatrix}$$

The design of the generalized functional observer is done as stated in previous chapters, the functional $z(t)$ must guarantee that the pair $(\bar{E}, \bar{A})$ is stable.

4.3.2 Numerical example

For this example, let the system of the form (4.24) with

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} -3 & 2 & 1 & 1 \\ 1 & -2 & 1 & -2 \\ 3 & 0 & 1 & -0.5 \\ -1 & -2 & 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0.5 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

be prone to actuator faults. The state feedback controller is calculated as

$$L = \begin{bmatrix} -8.95 & -26.07 & 12.1 & -4.85 \\ 1.71 & 1.31 & -3.51 & -0.16 \end{bmatrix}$$

For the augmented state, we have

$$\bar{L} = \begin{bmatrix} -8.95 & -267 & 12.1 & -4.85 & -1 & 0 \\ 1.71 & 1.31 & -3.51 & -0.16 & 0 & -1 \end{bmatrix}$$

In this case we chose matrix R as

$$R = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

It can be seen that the order of the observer is different from the functional $z(t)$. The parameter matrices are

$$N = \begin{bmatrix} 1.04 & 8.32 & 11.95 \\ -21.76 & -34.94 & -43.03 \\ 12.09 & 16.63 & 18.9 \end{bmatrix}, J = \begin{bmatrix} 0.05 & 0.05 & 0.05 \\ -0.17 & -0.17 & -0.17 \\ 0.1 & 0.1 & 0.1 \end{bmatrix},$$

$$F = \begin{bmatrix} 501.43 & 1.66 & -380.96 & 190.53 \\ -1815.274 & -48.76 & 1363.92 & -754.63 \\ 1008.47 & 18.97 & -760.63 & 405.41 \end{bmatrix}, H = \begin{bmatrix} 13 & 9.36 \\ -64.79 & -56.7 \\ 30.99 & 28.72 \end{bmatrix},$$

$$S = \begin{bmatrix} -0.0611 & -0.0278 & -0.0278 \\ -0.1110 & -0.0222 & -0.0111 \\ 0.0722 & 0.0361 & 0.0444 \end{bmatrix}, G = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}, M = \begin{bmatrix} 0.03 & 0.03 & 0.06 & -0.03 \\ 0.03 & 0.03 & 0.06 & -0.03 \\ 0.06 & 0.06 & 0.11 & -0.06 \end{bmatrix},$$

$$P = \begin{bmatrix} -12.1 & 12.1 & 11.1 \\ 2.51 & -3.51 & -2.51 \end{bmatrix}, \text{ and } Q = \begin{bmatrix} -402.34 & 432.42 & -1323.51 & 397.5 \\ 122.75 & -134.83 & 400.78 & -122.91 \end{bmatrix}.$$

Simulation

The simulation of the FTC system is done using Simulink. The initial values of the system are $x(0) = \begin{bmatrix} 1 & -1 & 3 & 1 \end{bmatrix}^T$. To illustrate the fault mitigation, the states of the FTC will be compared with a healthy reference model $x_d(t)$. Its dynamics are described by

$$\begin{aligned} E\dot{x}_d(t) &= Ax_d(t) + Bu_h(t) \\ u_h(t) &= Lx_d(t) + u_r(t) \end{aligned} \tag{4.27}$$

The actuator faults are given in Figure 4.10

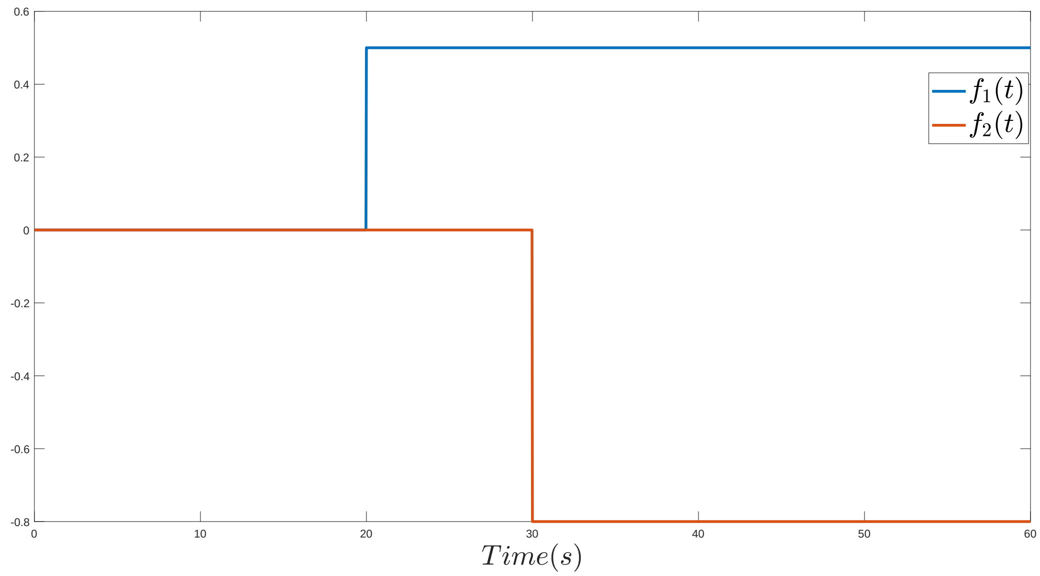


Figure 4.10: Faults in the system $f(t)$

The desired control law, and the comparison with the estimated fault-tolerant control law is illustrated in Figure 4.11

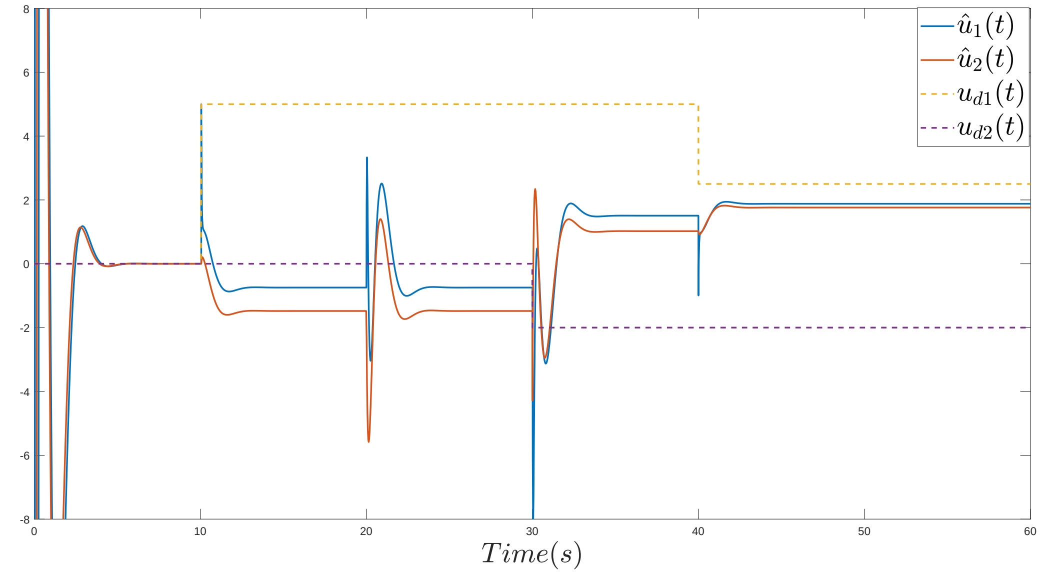
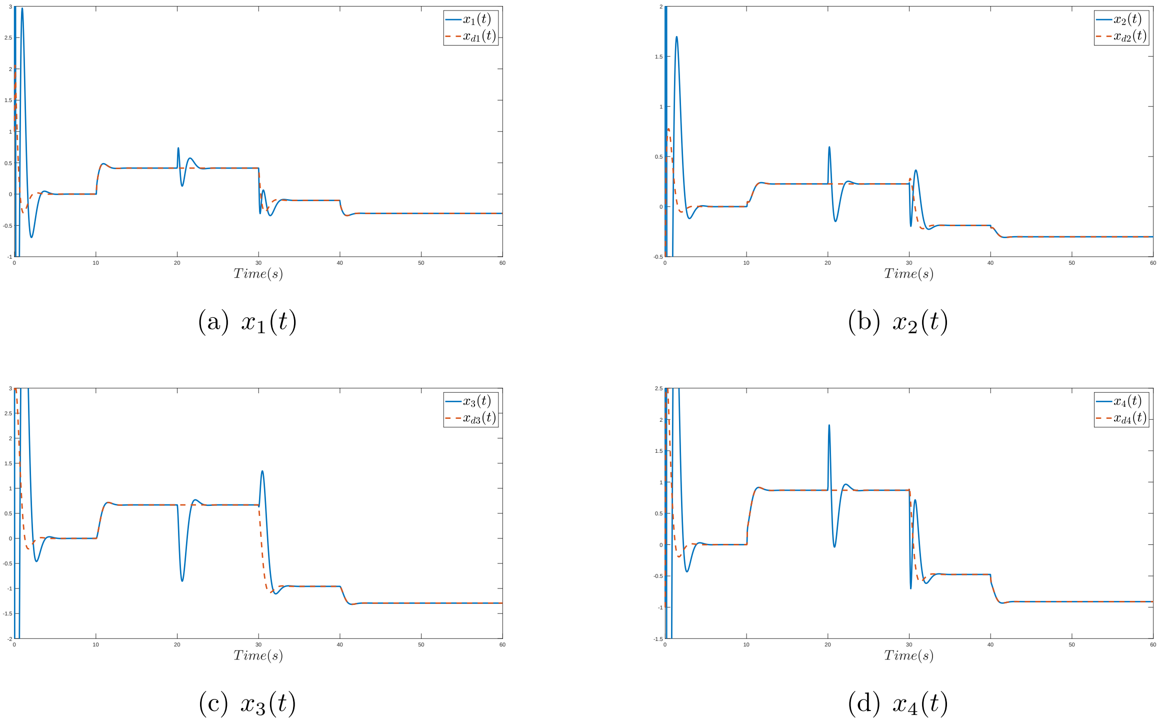


Figure 4.11: $u_r(t)$ and $\hat{u}(t)$

It can be seen that the real control law differs from the desired one. This is due to the observer compensating the controlled input in order to adjust the behavior of the system to the fault.

The results are given in Figure 4.12



(a) $x_1(t)$ (b) $x_2(t)$

(c) $x_3(t)$ (d) $x_4(t)$

Figure 4.12: Comparison between $x(t)$ and $x_d(t)$

Figure 4.12 shows that the FTC mitigates the effects of the actuator faults. It can be seen that the faulty system behaves similar to a healthy reference one.

4.4 Simultaneous state estimation and fault-tolerant control

One of the main advantages of the functional observers is its flexibility on the estimate $z(t)$. This flexibility opens up possibilities for achieving simultaneous fault-tolerant stabilization, as well as fault diagnosis and isolation. By exploiting this feature, engineers can design control systems that not only ensure system stability and performance in the presence of faults but also provide valuable insights into fault detection, identification, and isolation.

4.4.1 Problem statement

For a descriptor LTI system of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + F_a f(t) \\ y(t) &= Cx(t) + F_s f(t) \end{aligned} \tag{4.28}$$

where $f(t) = \begin{bmatrix} f_a(t) & f_s(t) \end{bmatrix}^T \in \mathbb{R}^{n_f}$ is the fault vector, $f_a(t)$ and $f_s(t)$ are the actuator and sensor faults respectively. The stabilizing FT control law is $u(t) = Lx(t) - f_a(t) + u_r(t)$ and will be estimated using a generalized functional observer of the form (4.3).

The firs step is to design a matrix L that makes the pair $(E, A + BL)$ stable. This method has been discussed previously in this chapter.

The system (4.28) can be rewritten as the following augmented system:

$$\begin{aligned} \bar{E}\dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}\hat{u}(t) \\ y(t) &= \bar{C}\bar{x}(t) \\ z(t) &= \bar{L}\bar{x}(t) \end{aligned} \tag{4.29}$$

where

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ f(t) \end{bmatrix},\ \bar{E} = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix},\ \bar{A} = \begin{bmatrix} A & F_a \\ 0 & 0 \end{bmatrix},\ \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix},\ \bar{C} = \begin{bmatrix} C & F_s \end{bmatrix}, \text{ and } \bar{L} = \begin{bmatrix} L & \begin{bmatrix} -I_a & 0 \end{bmatrix} \\ 0 & I_{n_f} \end{bmatrix}$$

4.4.2 Numerical example

For this example, let's consider a system with sensor faults. The objective is to estimate a fault-tolerant control law able to stabilize the system as well as estimate the occurring faults. The system of the form (4.21) with.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} -3.0 & 2.0 & 1.0 \\ -2.0 & 1.0 & -2.0 \\ -1.0 & -2.0 & 1.0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0.5 & -1 \\ 0 & 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $F_s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The stabilizing control law is defined as $L = \begin{bmatrix} 2.61 & 1.09 & -1.38 \\ -0.43 & 0.04 & 0.02 \end{bmatrix}$, and the augmented $\bar{L}$ is

$$\bar{L} = \begin{bmatrix} 2.61 & 1.09 & -1.38 & -1 & 0 \\ -0.43 & 0.04 & 0.02 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The parameter matrices are

$$N = \begin{bmatrix} -4.63 & 0.64 & 0.47 & -0.28 \\ 0.53 & -4.09 & 0.68 & -0.4 \\ 0.49 & 0.85 & -4.37 & -0.37 \\ 0.21 & 0.37 & 0.28 & -5.16 \end{bmatrix}, J = \begin{bmatrix} 0.71 & 0.71 & 0.71 & 0.71 \\ 1.01 & 1.01 & 1.01 & 1.01 \\ 0.94 & 0.94 & 0.94 & 0.94 \\ 0.41 & 0.41 & 0.41 & 0.41 \end{bmatrix},$$

$$F = \begin{bmatrix} 3.43 & -0.53 & -5.05 \\ -1.37 & -1.35 & 1.04 \\ -3.41 & -3.86 & 2.27 \\ -4.32 & -10.23 & -0.45 \end{bmatrix}, \; H = \begin{bmatrix} -2.53 & 0.36 \\ 0.52 & -0.34 \\ 1.14 & -0.91 \\ -0.23 & -1.82 \end{bmatrix}$$

$$S = \begin{bmatrix} 0.08 & 0.17 & 0.09 & -0.06 \\ -0.04 & -0.37 & 0.07 & 0.02 \\ 0 & -0.01 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; G = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.4282 & -0.4969 & -0.3055 \\ 0.0781 & -0.5188 & -0.1316 \\ -0.7141 & 0.2415 & 0.9595 \\ 0 & 0 & 0 \end{bmatrix} \; P = \begin{bmatrix} -0.88 & 0.16 & 0.17 & -0.09 \\ 0.16 & -0.08 & -0.09 & -0.1 \\ 0.17 & -0.09 & -0.61 & -0.14 \\ -0.09 & -0.1 & -0.14 & -0.93 \end{bmatrix},$$

$$\text{and } Q = \begin{bmatrix} -0.37 & -0.11 & -1.01 \\ -0.19 & -0.27 & 0.21 \\ -0.45 & 0.23 & 0.45 \\ -0.91 & -2.05 & 0.91 \end{bmatrix}$$

Simulation

The simulation of the FTC system is done using Simulink. The initial values of the system are $x(0) = \begin{bmatrix}1 & -1 & -1\end{bmatrix}^T$. The fault and its estimate is illustrated in Figure 4.13

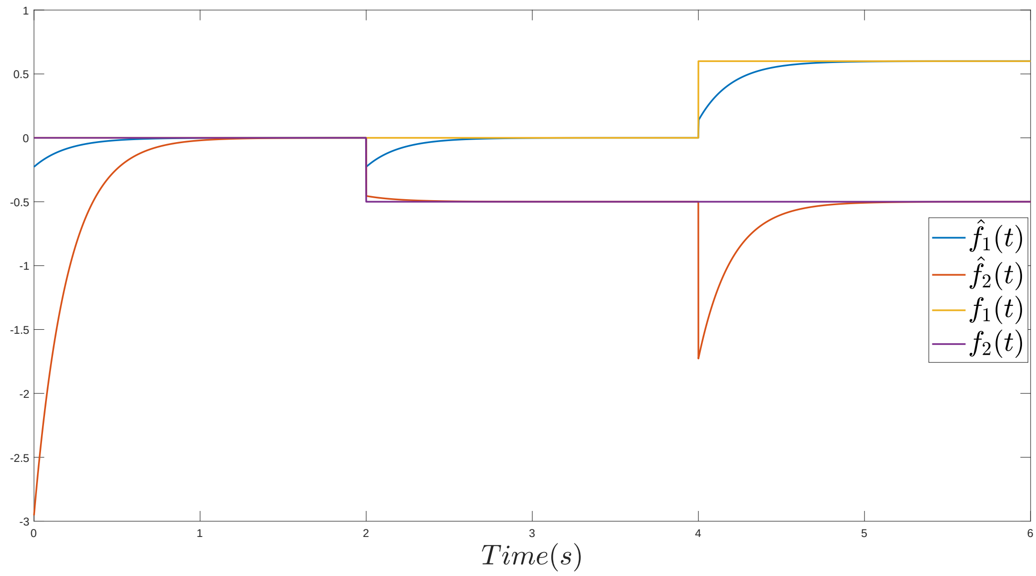


Figure 4.13: Estimated $\hat{f}(t)$ and its comparison with $f(t)$

It can be seen from Figure 4.13 that the faults are correctly estimated. Fault estimation

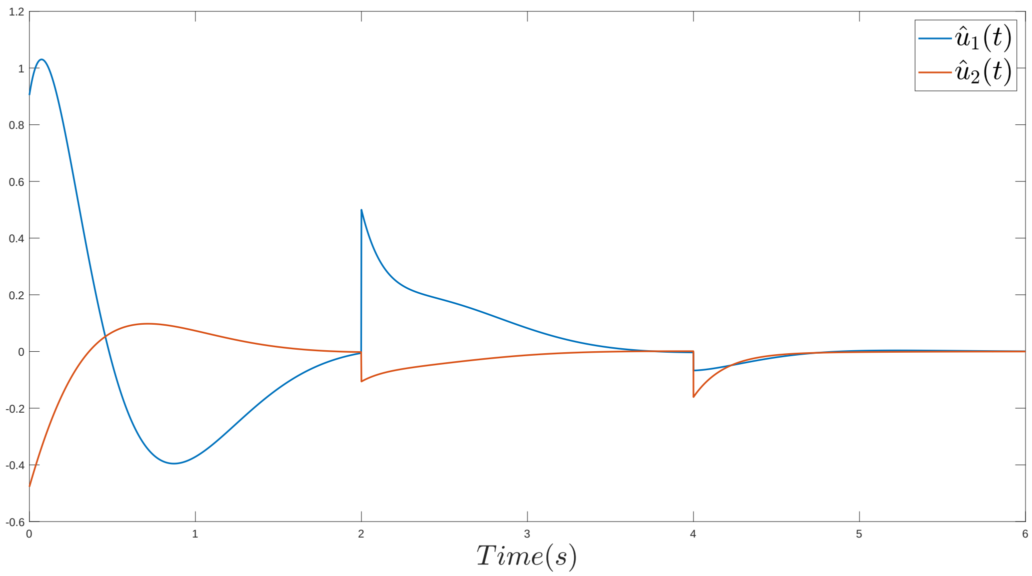


Figure 4.14: Estimated control law

contributes to improving the safety of control systems. By promptly detecting and diagnosing faults, potential safety hazards can be mitigated or avoided altogether. This is particularly critical in safety-critical applications.

Early fault detection and diagnosis facilitated by fault estimation can lead to reduced system downtime and maintenance costs. By detecting faults in their early stages, maintenance and repair activities can be scheduled proactively, minimizing unexpected system failures and reducing costly downtime. Additionally, fault estimation can aid in optimizing maintenance efforts by providing insights into the severity and urgency of the faults.

The fault-tolerant control law presented in Figure 4.14 offers an effective and robust solution for stabilizing unstable LTI descriptor systems in the presence of faults. Its integration in fault estimation techniques and ability to maintain stability and performance highlight its significance in fault-tolerant control systems.

Figure 4.15 illustrates the stable states of the system. It can be seen that the behavior of the system remains consistent and within desired bounds, indicating the effectiveness and reliability of the fault-tolerant control approach.

In conclusion, this example has presented a comprehensive investigation of a fault-tolerant control law for stabilizing an unstable LTI descriptor system in the presence of sensor faults. The proposed fault-tolerant control law has demonstrated its effectiveness in achieving system stability and maintaining desired performance objectives even in the presence of faults. Through the integration of fault estimation techniques, the control law provides timely fault detection, diagnosis, and isolation, contributing to the overall fault tolerance capabilities of the system.

The simulation results have shown that the fault-tolerant control law successfully compen-

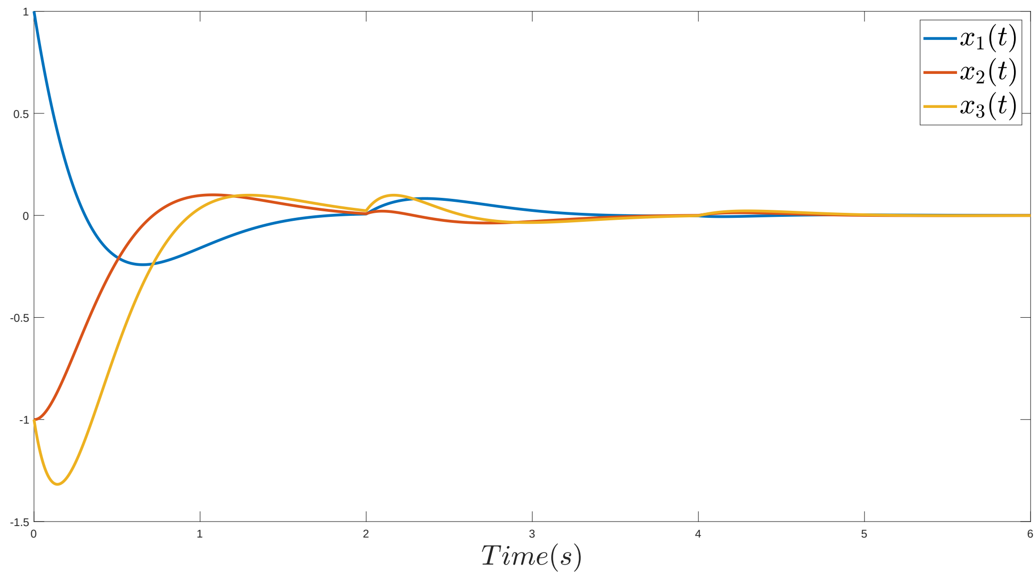


Figure 4.15: States of the system

sates for the effects of sensor faults, ensuring the stability and performance of the system. It effectively detects and isolates faults, allowing for appropriate corrective actions to be taken in a timely manner. The control law has exhibited robustness and resilience against different fault scenarios, accommodating variations in fault magnitudes, durations, and patterns.

4.5 Conclusions

This chapter has provided a comprehensive exploration of observer-based control within the context of functional observers for linear descriptor systems. The fundamental principles, design methodologies, and practical applications of observer-based control have been thoroughly discussed, highlighting its crucial role in achieving precise regulation and manipulation of dynamic processes.

Through the exploration of functional observers, we have gained a comprehensive understanding of their design methodologies, including the necessary and sufficient conditions for their existence. The insights gained from this exploration contribute to the body of knowledge on observer-based control, empowering engineers to design state estimators that accurately estimate the internal states of a system, leading to the development of robust, responsive, and versatile control strategies.

Moreover, the integration of fault-tolerant control within the framework of observer-based control has been addressed. Fault tolerance is essential for ensuring system robustness and resilience in the face of faults that may arise during operation. By incorporating fault detection, identification, and accommodation techniques with observer-based control, simultaneous fault-tolerant control and state estimation can be achieved, enhancing the overall performance and reliability of control systems.

The case studies and simulations presented in this chapter have demonstrated the effectiveness of the proposed observer-based fault-tolerant control approach for linear descriptor systems. The results have showcased improved system performance, robustness against faults, and the ability to maintain accurate state estimation even in the presence of faults or uncertainties.

By leveraging functional observers, engineers can design control systems that are not only robust and resilient to faults but also capable of accurately estimating system states, leading to improved performance and reliability. The advancements made in this area have the potential to positively impact various industries, including aerospace, automotive, robotics, and process control, among others.

As we move forward, it is essential to continue exploring, refining, and expanding upon the concepts and techniques discussed in this chapter. By doing so, we can advance the state-of-the-art in observer-based control and fault-tolerant control, bringing us closer to achieving more reliable and robust control systems in the face of uncertainties and faults.

Chapter 5

Conclusions

Contents

The work presented in this thesis contributes to the estimation, control, and fault diagnosis for Linear Descriptor Systems. Functional observers have a good potential in control theory, their simplicity allows us to design fast and reliable control systems and use them in combination with other tools to create powerful diagnostic and fault-tolerant systems.

In Chapter 3 a method for the synthesis of Finite-Time Functional Observers for continuous-time descriptor systems was presented. The existence conditions were given and the stability conditions were set in terms of LMI approach. D-stability was chosen to guarantee the finite-time convergence of the observer. This Chapter also explores the non-linear systems and proposes a method for the synthesis of a Takagi-Sugeno functional observer. In this chapter we concluded that the linear theory of the functional observers proposed can be used to estimate the state of Takagi-Sugeno systems. This TS systems can be used to represent non-linear behaviors of a plant. Finally, in Chapter 3, unknown inputs are presented in linear systems, and are estimated by using a proposed functional observer.

In Chapter 4, the focus is on the problem of observer-based control for descriptor linear time-invariant (LTI) systems. The chapter delves into the synthesis of fault-tolerant control strategies specifically designed to address sensor faults, actuator faults, and the simultaneous combination of fault-tolerant control and fault estimation. The developed fault-tolerant control methodologies offer effective solutions for mitigating the effects of sensor faults, actuator faults, and the simultaneous occurrence of faults while ensuring the stability and desired performance of the system. The integration of fault estimation techniques within the control framework contributes to the fault tolerance capabilities, enabling timely detection, diagnosis, and accommodation of faults.

In this final chapter, we summarize the key findings, contributions, and implications of the research conducted in this thesis. We revisit the main objectives outlined in the introduction and reflect on the insights gained throughout the study. Additionally, we discuss the limitations of the research and propose avenues for future research in the field.

5.1 Contributions

This research has made significant contributions to the field of functional observers and control for descriptor systems. The following key contributions have been achieved:

1. Development of Finite-Time Convergence Generalized Functional Observers:
   - We have proposed and investigated the novel concept of finite-time convergence generalized functional observers.
   - This contribution extends the traditional functional observer framework by ensuring convergence within a finite time horizon, providing a valuable tool for time-critical applications.
2. Functional Observers for Takagi-Sugeno Descriptor Systems:
   - We have developed functional observers specifically tailored for Takagi-Sugeno descriptor systems, which are widely used in modeling complex systems with nonlinear dynamics.
   - The proposed observers offer a systematic and effective approach for state estimation in Takagi-Sugeno descriptor systems, considering both continuous and discrete modes.
3. Generalized Dynamical Observer with Unknown Inputs:
   - We have introduced the concept of the generalized dynamical observer with unknown inputs, which enables the estimation of both system states and unknown inputs.
   - This contribution enhances the observer's capability to handle unmeasured inputs, improving the accuracy of state estimation and fault detection in practical systems.
4. Functional Observer-Based Control for Descriptor Systems:
   - We have explored the integration of functional observers into the control framework for descriptor systems.
   - By combining state estimation and control in a unified framework, our research provides a comprehensive solution for achieving robust and efficient control in the presence of uncertainties and disturbances.

These contributions advance the state-of-the-art in functional observers and control for descriptor systems, offering novel methodologies and techniques to address challenging real-world problems. The proposed approaches extend the existing theoretical foundations and provide practical solutions that have potential applications in diverse fields.

Moreover, the contributions of this research offer practical implications for system design, fault diagnosis, and fault-tolerant control strategies. The developed methodologies enhance system performance, reliability, and fault tolerance, leading to improved operational efficiency and reduced downtime in critical applications.

5.2 Products

As results for this research study, four scientific articles have been accepted and presented; two in national congresses, with titled "**Generalized Functional Observer Design for Descriptor Linear Systems**" (In Congreso Nacional de Control Automático 2017, October 2017, Monterrey, Mexico) and "**Synthèse d'observateurs fonctionnels à temps de convergence finie**" (In Congrès National des IUT, June 2019, Toulon, France).

And two articles in international congresses, titled "**Finite time functional observers for descriptor systems. Application to fault tolerant control**". (In 2019 27^{th} Mediterranean Conference on Control and Automation (MED)), and "**Design of finite-time convergent functional observers**" (In 2019 8^{th} International Conference on Systems and Control, October 2019, Marrakesh, Morroco).

The presented results where used to publish a scientific article in the magazine *Processes* (doi: 10.3390/pr11061707) with the title **Generalized Functional Observer for Descriptor Nonlinear Systems—A Takagi-Sugeno Approach**.

5.3 Discussion

In this section, we engage in a comprehensive discussion of the research findings, their significance, and their implications for the field of functional observers and control for descriptor systems. We also address any limitations and challenges encountered during the research process.

The key findings of this research significantly contribute to the field of functional observers and control for descriptor systems. The development of finite-time convergence generalized functional observers ensures convergence within a finite time horizon, addressing time-critical applications and advancing the field's knowledge. The tailored functional observers for Takagi-Sugeno descriptor systems enable state estimation in complex systems with nonlinear dynamics, enhancing the applicability of functional observers in practical engineering domains. The integration of functional observers into the control framework for descriptor systems provides a unified approach that combines state estimation and control, leading to improved performance, fault tolerance, and adaptability. These findings align with the original research objectives outlined in the introduction and offer novel methodologies, enhanced performance, and practical implications for real-world applications. Overall, these key findings contribute to the advancement of knowledge and understanding in the field, extending the capabilities of functional observers and control strategies for descriptor systems.

The research findings have significant implications for real-world applications in various industrial systems and engineering domains. The developed methodologies and techniques offer practical solutions for improving system performance, reliability, and fault tolerance in practical scenarios.

The implementation of finite-time convergence generalized functional observers in time-critical applications can have a profound impact. Industries such as aerospace, robotics, and autonomous vehicles can benefit from the real-time estimation capabilities of these observers, enabling faster and more accurate decision-making. The ability to ensure convergence within a finite time horizon enhances system stability, safety, and operational efficiency in time-constrained scenarios.

The tailored functional observers for Takagi-Sugeno descriptor systems have direct implications for industries dealing with complex systems exhibiting nonlinear dynamics. By accurately estimating system states, these observers facilitate system monitoring, fault diagnosis, and control in domains such as process industries, power systems, and chemical engineering. The robust estimation capabilities of these observers enable precise modeling and control, improving system performance and reliability.

The integration of functional observers into the control framework for descriptor systems offers practical benefits for enhancing system performance, reliability, and fault tolerance. Industries with safety-critical applications, such as automotive systems, industrial automation, and renewable energy systems, can leverage these techniques to ensure uninterrupted operation and improve system resilience. By combining state estimation and control, these approaches enable proactive fault management and adaptive control actions, mitigating the effects of faults and uncertainties and enhancing overall system reliability.

Overall, the research findings have practical implications across a range of industrial systems and engineering domains. The developed methodologies and techniques provide practical solutions to improve system performance, reliability, and fault tolerance. Their implementation can lead to enhanced operational efficiency, reduced downtime, improved safety, and increased system resilience in real-world applications.

5.4 Limitations

It is important to acknowledge the limitations and constraints of this work. These limitations provide opportunities for future research and areas of improvement.

A limitation of this work is its reliance on simplified models and assumptions, which were necessary to facilitate the development of control strategies and observer designs. Certain assumptions were made regarding system dynamics, noise characteristics, and fault patterns to enable theoretical advancements. However, it is important to acknowledge that these simplified models and assumptions may not fully capture the complexity and variability of real-world systems. While they provided a foundation for theoretical developments, future research should focus on considering more realistic and complex models to enhance the practical applicability of the developed methodologies. Incorporating more intricate and diverse system dynamics, as well as realistic noise characteristics and fault patterns, will allow for a more comprehensive assessment of the proposed approaches and ensure their effectiveness in real-world applications.

A further consideration lies in the practical implementation of the proposed methodologies and their robustness to uncertainties. Real-world systems commonly exhibit unknown or time-varying parameters, uncertainties, and disturbances that can significantly impact the performance of functional observers and control strategies. While the proposed methodologies show promise in controlled environments, their practical implementation may face additional challenges. To overcome this limitation, future research should prioritize the development of robust methodologies that can effectively handle uncertainties, adapt to changing conditions, and ensure reliable system operation under realistic scenarios. By incorporating robustness considerations and accounting for uncertainties, the proposed methodologies can enhance their practical applicability and provide more reliable and resilient control solutions for real-world systems.

An important consideration is the limited focus on experimental validation and real-world case studies in this work. While the research has primarily concentrated on theoretical developments and simulation-based evaluations, it is crucial to supplement these findings with experimental validation to fully assess the practical effectiveness and robustness of the proposed methodologies. Conducting experiments in realistic environments and collaborating with industry partners can provide valuable insights and validation of the developed techniques. By incorporating experimental validation and real-world case studies, researchers can gain a deeper understanding of the practical implications and limitations of the proposed methodologies, ensuring their applicability in real-world scenarios and validating their performance under varying conditions. Further emphasis on experimental validation and real-world case studies will enhance the credibility and practical relevance of the research outcomes.

This research has addressed key challenges and proposed methodologies to enhance the resilience, performance, and practical applicability of these systems. However, certain limitations, such as the reliance on simplified models and assumptions, the need for robustness to uncertainties, and the importance of experimental validation and real-world case studies, have been identified. Future research should focus on addressing these limitations and further advancing the proposed methodologies. By considering more realistic and complex models, developing robust methodologies, and conducting experimental validations, the proposed approaches can be refined and their effectiveness in real-world scenarios can be thoroughly assessed. These efforts will contribute to the development of more reliable, adaptable, and practical solutions for functional observers and control in descriptor systems.

5.5 Future works

While this research has made significant contributions to the field of functional observers and control for descriptor systems, several promising avenues for future research and development emerge. Specifically, future works should focus on addressing the challenges posed by actuator and sensor saturation, as well as extending the proposed methodologies to handle nonlinear systems, and considering the effects of noise in output measurements.

1. **Actuator and Sensor Saturation:**
    - Future research should focus on incorporating actuator and sensor saturation into the control framework for descriptor systems. Considering the impact of saturation on system stability and performance is crucial for practical applications. Developing control strategies that explicitly account for actuator saturation and propose appropriate saturation mitigation techniques will be valuable.
    - Similarly, addressing the challenges posed by sensor saturation is important for reliable state estimation and fault detection. Investigating robust observer-based approaches that can handle sensor saturation and mitigate its effects on state estimation accuracy will improve the overall system's performance.
2. **Extension to Nonlinear Systems:**
    - Extending the proposed methodologies to handle nonlinear systems is an important direction for future research. Many real-world systems exhibit nonlinear behavior, and incorporating nonlinear dynamics into the observer design and control strategies is essential.
    - Investigating adaptive functional observers or nonlinear observer-based control methods can offer promising solutions for nonlinear systems. These techniques can adaptively estimate the nonlinear dynamics and handle uncertainties, improving the accuracy and robustness of state estimation and control.
3. **Handling the Effects of Noise in Output Measurement:**
    - In practical systems, output measurements are often corrupted by noise, which can impact the accuracy of state estimation and fault detection. Future research should focus on developing robust methodologies that explicitly account for the effects of noise in output measurements.
    - Investigating noise-robust observer design techniques, such as filtering or estimation algorithms that can effectively mitigate the impact of noise, will enhance the accuracy and reliability of state estimation.
4. **Validation and Real-World Implementation:**
    - Validating and implementing the proposed methodologies in real-world scenarios is essential. Conducting extensive experiments, case studies, or industrial collaborations will provide valuable insights into the performance and practical applicability of the developed techniques, considering the effects of measurement noise.

- Additionally, investigating the scalability and adaptability of the proposed methodologies to large-scale systems or different application domains, while accounting for measurement noise, will be crucial for their wider adoption and successful implementation.

By addressing the challenges of actuator and sensor saturation, extending the methodologies to handle nonlinear systems, integrating fault diagnosis and fault-tolerant control while considering the effects of noise in output measurements, and validating the approaches in real-world scenarios, future research can further enhance the capabilities and practical applicability of functional observers and control for descriptor systems. These areas offer exciting opportunities for advancing the field and addressing the complex challenges in system control, fault tolerance, and the presence of measurement noise.

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